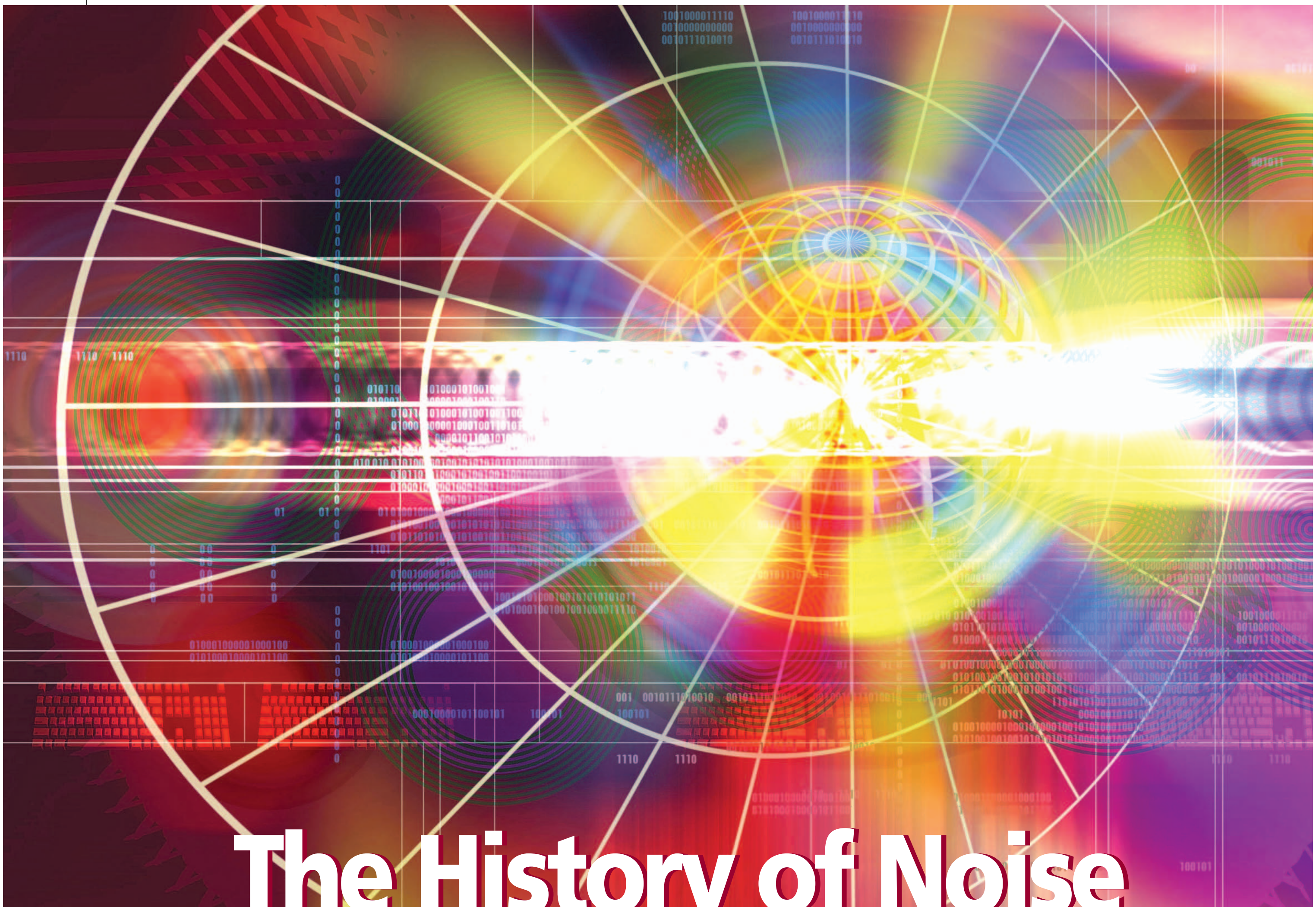

The History of Noise

[On the 100th anniversary of its birth]

[Leon Cohen]

Noise had a glorious birth. It was one of the three miracles of the miracle year, 1905. Einstein, always aiming to solve the greatest of problems and to solve them simply, saw that noise could be the instrument to establish one of the greatest ideas of all time—the existence of atoms. In a few simple pages he invented noise, and thus "noise" was born. Immediately after Einstein, there was an incredible flurry of ideas of the most profound kind which continues to this day. Noise permeates every field of science and technology and has been instrumental in solving great problems, including the origin of the universe. But noise, considered by many as unwanted and mistakenly defined as such by some, has little respectability. The term itself conjures up images of rejection. Yet it is an idea that has served mankind in the most profound ways. It would be a dull, gray world without noise. The story of noise is fascinating, and while in its early stages, noise's story was clearly told, its subsequent divergence into many subfields has often resulted in a lack of understanding of its historical origins, development, and importance. We try to give it some justice. We discuss who did what, when, and why, and the historical misconceptions. But most importantly, we aim to show that the story of noise is an exciting story, filled with drama, and worth telling.

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INTRODUCTION TO NOISE

"Noise," as an idea, a subject, a field, an instrument, came upon the scene with a power and swiftness that transformed all of science and our views of the nature of matter. At birth, it solved the major issue of its time, perhaps, the greatest idea of all time—the existence of atoms. The debate on the reality of atoms had reached a crescendo. The debaters were the greatest of scientists; there was no middle ground, either atoms exist or they do not. The bitterness of the atomists and anti-atomists got extreme, and while no one dreamed of seeing an atom, everyone knew they were debating the greatest of issues:

> "If . . . all of scientific knowledge were to be destroyed, and only one sentence passed on to the next generations of creatures . . . it is . . . all things are made of atoms."
>
> —Richard Feynman

As it turned out, not only do atoms exist, but they are the most exquisite creation of nature; neither the solar system, galaxies, nor anything else can rival the atom's simple complexity. The 19th century, the century of great achievements—thermodynamics, electromagnetism, chemistry, and the industrial revolution—did not need the "atom." Yet, it was the century of the atomist debate, a debate that raged into the beginning of the 20th century; until Einstein, always aiming at the greatest of problems, and aiming to solve them simply, saw the instrument to prove their existence! In a few simple pages he invented noise and thus "noise" was born. This was in 1905. Things moved quickly. Within a few years, Perrin verified Einstein's main prediction and also his prediction that noise could be used to calculate Avogadro's number! Avogadro, who came up with one of the most profound ideas of all time, died without any recognition, never dreamed that there would be a number named after him, and certainly he, nor anyone else, could have imagined that noise would be the instrument for its calculation and for the awarding of a Nobel Prize.

It was the end of the anti-atomists but the beginning of the proud history of noise. Immediately after Einstein, there was a flurry of ideas of the most profound kind that continues to this day. Within three years, Langevin started the field of stochastic differential equations, although that was not his motivation. There were numerous important contributions that laid both the foundation of "noise" and its application to many fields. The historical twists are fascinating. Who could have imagined that the search for atmospheric noise would lead to the discovery of the noise at the origin of the universe and establish the "big bang" theory of the universe?

But noise, considered by many as unwanted, and mistakenly defined as such by some, has little respectability. The term conjures up images of rejection, images of building filters to eliminate it. Yet it is an idea that has served mankind in the most profound ways. It would, indeed, be a dreary world without noise.

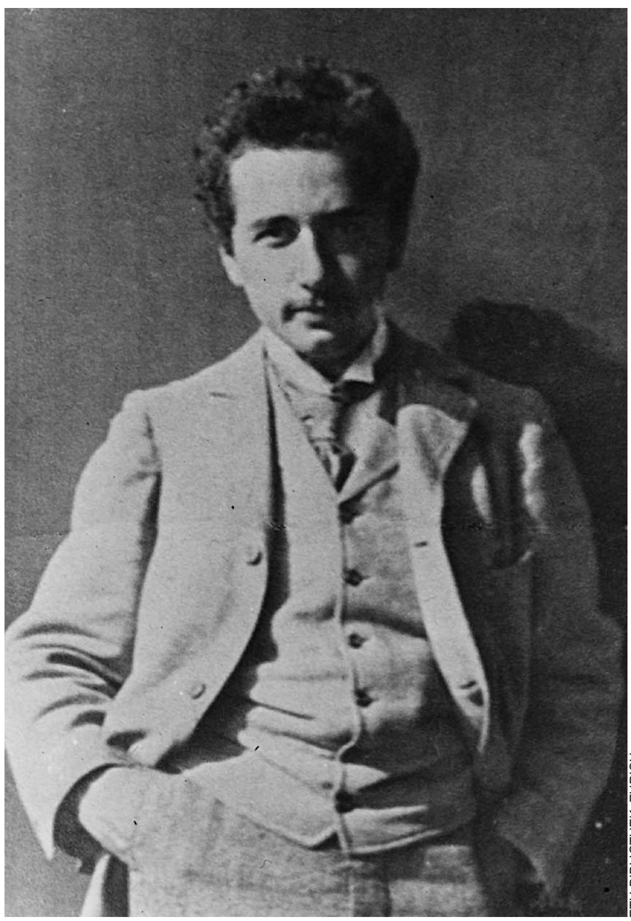
It is now 100 years since Einstein devised "noise." Noise permeates every field of science, and every field seems to have its own version of its history. While some fields tell it almost properly, most don't. The often told version—that Brown discovered, Einstein explained, Langevin simplified, and Perrin verified—is a serious historical distortion. But more importantly, it leaves out the drama and excitement of the story. The story of noise is a fascinating one, but its divergence into many subfields has often resulted in a lack of understanding of noise's true historical development. We try to give it some justice and discuss who were the main players, who did what, when, and why, and the reasons for the impact on so many fields. But more importantly, we aim to show that the history of noise is a tale worth telling. We hope, though, that we do not say any more than most readers want to know.

HISTORY AND SCIENCE HISTORY

Almost every school child since the dawn of school has hated the study of standard "history." Rightfully so, since it is as boring as things can be. In one way or another, we are told that history is important. And in one way or another, we are told, as Santayana put it, "Those who cannot learn from history are doomed to repeat it." This is certainly one of the silliest things ever said. Presumably, what is lamented here are the horrible things like wars that we don't wish to repeat, yet since the dawn of history almost all leaders were given the finest of schooling in history. If anything, the leaders learned history so that they could repeat it. To be fair to Santayana, he also said “History is a pack of lies about events that never happened told by people who weren't there.” It would be a new world, perhaps even a braver one, if we admitted the obvious, as expressed by Huxley, “That men do not learn very much from the lessons of history is the most important of all the lessons of history.”

[EINSTEIN, ALWAYS AIMING AT THE GREATEST OF PROBLEMS, AND AIMING TO SOLVE THEM SIMPLY, SAW THAT NOISE COULD BE THE INSTRUMENT TO ESTABLISH ONE OF THE GREATEST IDEAS OF ALL TIME, THE EXISTENCE OF ATOMS.]

On the other hand, science history is exciting and inspiring. Moreover, it is a great way to learn science. It is truly fascinating to learn how the greatest of minds came up with the greatest of ideas, and that makes science history entertaining. Moreover, it is a fact that if one is trying to learn an idea, the originator is the place to go. It is often much more instructive to read the original papers on a subject than to learn it from a textbook. If the original author is a clear writer, which is often the case with great scientists and mathematicians, we see the simplicity of their arguments, motivations, and reasons much clearer than in subsequent presentations. This is particularly so in the case of noise. If one, for example, wants to get an idea of what stochastic differential equations are all about, the original papers of Langevin, Ornstein, Uhlenbeck, and Chandrasekhar are worth hundreds of current books on the subject. Moreover, in the case of noise, we have Einstein, one of the simplest and clearest writers ever. The introductions, or just the first paragraphs, of his papers or writings are simple, clear, powerful, and fascinating to read. Simply reading them is an incredible education because he gets to the essence of the subject with remarkable simplicity and clarity. Fortunately, he left a voluminous amount of writings on a wide variety of subjects.



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[FIG1] **Einstein as a young man.**

EINSTEIN: WHY HE DID IT

Einstein is popularly imagined as a demigod who never changed his clothing, whose sweaters had holes, who was always immersed in deep thought so advanced that no one could understand him, who was always right about everything, and whose photographs in newspapers always fostered that impression. Forget that image and forget that Einstein was interested in explaining Brownian motion, the erratic movement of pollen and dust. Einstein was straightforward, direct, always clear, ambitious in an honorable sense, and, for whatever reasons, conscious or subconscious, decided that he would tackle the biggest problems and that he would attack them simply. Look at a picture of Einstein as a young man, and you will not see the mythical Einstein (Figure 1). Read his writings, and you will see the simplicity of his motives and desires. And certainly his letters, and his love letters are of a man who knew what he wanted.

So, again, forget the often-stated notion that Einstein wanted to explain the erratic movement of pollen in water. He was after proving that atoms exist! Moreover, he went after the ultimate method that defines greatness in science. Predict an effect, derive a specific formula, let the world perform the experiments, and there you have it. Einstein searched for a manifestation of these invisible atoms that could be seen and measured. So Einstein said, if atoms exist, then I predict an effect and I derive a specific formula relating to the effect, and if this formula is verified, then . . . ! With the courage to say and derive it all in a few simple pages, he predicts a macroscopic manifestation of atoms. Within a few years, his prediction was proven true and changed the tide: everyone believed in atoms even though no one saw them.

It was 1905 and Einstein's first Brownian motion paper was one of the four papers that would constitute the three miracles of the miracle year [1]. The title says it all: “On the

Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of Heat." Another translation uses the word "required" instead of demanded. That is, if atoms exist, then small particles immersed in liquids *must* behave in a way to be described and, therefore, if the small particles do indeed behave this way, then atoms exist. The first sentence reads "In this paper it will be shown . . . bodies of microscopically visible size suspended in liquid will perform movement of such magnitude that they can be easily observed in a microscope on account of the molecular motions of heat." Einstein concludes the two-paragraph introduction with "If the movement discussed here can actually be observed . . . an exact determination of actual atomic dimensions is possible. On the other hand, if the prediction of this motion were to be proved wrong, a weighty argument would be provided against the molecular-kinetic theory of heat." He concludes the paper with ". . . the relation can be used for the determination of N; " N was previously defined as the yet unnamed Avogadro's number. The last sentence of the paper reads "It is hoped that some inquirer may succeed shortly in solving the problem posed here, which is so important in connection with the theory of heat." That inquirer would be Perrin.

There is a book series, put together by Paul A. Schilpp, on living philosophers and scientists. Each book consists of contributed articles, and the living philosopher/scientist gets to reply and comment. All the books in the series are great, and the one on Einstein is particularly so. Einstein wrote "autobiographical notes" in the beginning and a "reply to criticism" in volume two. He starts his autobiographical notes with "Here I sit in order to write, at the age of 67, something like my own obituary." He comments on his Brownian motion work, which we just described, and ends with "My major aim in this was to find facts which would guarantee the existence of atoms . . ."

ANNUS MIRABILIS: THE MIRACLE YEAR/ THE EXTRAORDINARY YEAR

The year 1905 is called the miracle year. Einstein was 26. The phrase Annus Mirabilis was traditionally applied to the "year" 1665/1666, when Newton revolutionized everything in such a sustained effort that perhaps miracle is too mild a phrase. Newton was 23. The year wasn't a year, but about 18 months, but one shouldn't quibble. It was the time of the plague, and at the end of 1665, Cambridge University closed officially and Newton went back to his home town. In the subsequent 18 months, he revolutionized science and mathematics, inventing mechanics, gravity, light, and calculus, among other subjects, with ideas and methods that seemed to come out of nowhere. Not nowhere, but out of Newton. Of course, he had been thinking about these things before, but at 23, he couldn't have been thinking about them for too long. It is no less the case with Einstein's miracle year, 1905, the year he was working at the patent office. Of course, he had been thinking about these ideas for quite a few years, but in his own words, "A storm broke loose in my mind."

There were three miracles. One of the three is what we now call Brownian motion, and the aim was to prove the existence of atoms. The second was the explanation of the photoelectric effect, where he introduced the idea of the photon, although the name photon was coined 21 years later (1926) by Lewis, one of the greatest chemists of the last century. The explanation of the photoelectric effect started a chain of events concerning the nature of light, and it was an instrumental idea that would develop into the new view of matter and light that we now call quantum mechanics. The third miracle was the special theory of relativity, which totally changed our view of space and time and has the consequence that energy and mass can be transformed into each other.

The actual number of papers Einstein published in 1905 is five. All were relatively short, simply written, with a clarity of purpose and style that would mark all his papers. Of the three miracles just mentioned, four of the five papers are directed to them. They are "On a Heuristic Point of View Concerning the Production and Transformation of Light" (photoelectric effect) [2], "On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat" (Brownian motion) [1], "On the Electrodynamics of Moving Bodies" (relativity) [3], and "Does the Inertia of a Body Depend upon Its Energy Content?" [4]. The answer to the last one is yes, and of course everyone in the world would get to know perhaps the most famous equation in history, $E = mc^2$, except perhaps for $F = ma$.

BROWNIAN MOTION

Einstein did mention Brownian motion in his first paper: "It is possible that the movements described here are identical with the so-called Brownian motion; however the information available to me . . . is so imprecise that I could not form a definite opinion on this matter." He began his second paper [5], published a year later (1906), by expressing his regrets: "Soon after the appearance of my paper . . . Siedentopf informed me that he and other physicists . . . Prof. Gouy . . . had been convinced by direct observation that the so-called Brownian motion is caused by the irregular thermal movement of the molecules of the liquid." We stress, though, it was Einstein who developed the statistical properties and got specific results. In the title of the second paper, he used the word "Brownian" in "On the Theory of Brownian Movement." He published a paper [6] in 1907 titled "Theoretical Observations on the Brownian Motion," in which he gives a review and expresses: "I hope I may be able by the following to facilitate for physicists who handle the subject experimentally the interpretation of their observations as well as the comparison of that latter with theory." Further, in 1908, he published "The Elementary Theory of the Brownian Motion" [7]. This paper begins with "Prof. R. Lorentz has called to my attention, in a verbal communication, that an elementary theory of the Brownian motion would be welcomed by a number of chemists." Einstein then develops the connection with diffusion in an explicit way. Incidentally, Lorentz was the greatest physicist of his time. So, after the first paper, the phrase "Brownian motion" became standard.

NOISE, STOCHASTIC PROCESSES AND THE EINSTEIN (WIENER-KHINTCHINE) THEOREM

Einstein published many papers on stochastic processes for years after his original papers on Brownian motion. While there are many reasons for this continued interest, fundamentally his interest in the nature of light, the blackbody spectrum, the so-called Einstein A and B coefficients, Bose-Einstein statistics, and statistical mechanics, among other issues, all required that he develop new methods regarding noise and stochastic processes. It would take a book to put Einstein's contributions to noise in proper perspective, but perhaps it is worthwhile to mention that what we now call the Wiener-Khintchine theorem, the relation between the autocorrelation function and the power spectrum, was originally done by Einstein in 1914, years before Wiener or Khintchine. It was done in two papers [8], [9] titled "A Method for the Statistical Use of Observations of Apparently Irregular, Quasiperiodic Process" and "Method for the Determination of Statistical Values of Observations Regarding Quantities Subject to Irregular Observations." That same year, Einstein published a number of papers on gravitation and relativity. Of course, 1914 is the year before the famous 1915 paper on general relativity. We previously mentioned how clear Einstein's first paragraphs always are; it is worthwhile to reproduce the first paragraph of one of the papers just mentioned as it is so relevant to his interest and contributions to signal processing: "Suppose that one observes quasiperiodicaly fluctuating quantity F as a function of an independent variable t for a very large t-interval T. How can one obtain statistical data of a perspicuous character concerning F for observation? In what follows I present a new kind of method by which to attain this goal." By the way, it seems that perhaps it is Einstein who first defined the autocorrelation function: "To this end we introduce a quantity $\chi(\Delta)$, which we call the 'characteristic' and which shall be defined as follows:

$$\chi(\Delta) = \overline{F(t)F(t+\Delta)} = \frac{1}{T}\int_0^\infty F(t)F(t+\Delta)dt. \qquad (1)$$

This equation appears in his notation. So besides originating the Wiener-Khinchin theorem, I think it is fair to say that Einstein also is the one who came up with the autocorrelation function. He added that "It will turn out that there exists a simple dependence between the characteristic and the intensity curve;" that is, between the autocorrelation function and the power spectrum. Einstein then concluded that (leaving out the constant of integration)

$$2\chi(\Delta) = \int_0^\infty I(x)\cos x\Delta dx, \qquad (2)$$

where $I(x)$ "shall be called the spectral intensity," which he derived from the Fourier series in the now usual way. Equation (1) is what is commonly called the Wiener-Khinchin theorem.

BROWN AND BROWNIAN MOTION

Brown did not discover Brownian motion, but he studied it seriously, systematically, exhaustively, and passionately. While everyone mentions his 1828 paper, rarely do people mention the title: "A Brief Account of Microscopical Observations Made in the Months of June, July, and August, 1827 on the Particles Contained in the Pollen of Plants; and on the General Existence of Active Molecules in Organic and Inorganic Bodies" [10]. Brief? The article is not brief! It is written in the first person, which was not an unusual way of writing at that time. It was common in papers to see phrases like "I did this," "I did that," and "I traveled here and there." Of course, the word "molecule" does not mean molecule in our sense, but it means a small thing. Brown was a famous botanist. At an early age, he gambled and went on an official ship expedition to Australia that lasted about two years. While there, he collected numerous plants that no one had studied before. After his return, he spent years studying these new plants, and for this he earned his reputation. When he wrote the famous Brownian motion paper, Brown was about 45 years old.

Brownian motion was discovered in the early days of the invention of the microscope. With the invention of the microscope came the discovery that a drop of pond water contains an incredible world of microscopic life, single-celled and multicelled organisms of incredible varieties. Most of the organisms moved about seemingly without rhyme or reason. In the hundreds of years since its invention, the microscope has kept many kids glued to it, mesmerizing them with an amazing world. Anyway, it was noticed that other things like pollen also had erratic movements, and yet they seemed very different than the obviously live paramecium, amoebas, and such. But could these erratic movements of pollen indicate that they are alive and kicking; could they be the most primitive life yet? Many thought that these erratic movements were possibly due to some primitive life force. Brown saw the huge stakes in answering this question and went all out. He started with the usual pollen and then went through an incredible number of materials that are obviously not alive. "Having found motion in the particles of the pollen of all the living plants which I examined, I was led next to inquire whether this property continued after death of the plant . . . either dried or immersed in spirits for a few days only, the particles of pollen . . . were found in motion equally evident with that observed in the living plants." Brown even studied dried plants "no less than a century" old. He also studied minerals and woods of all kinds, as well as anything else he could think of. Everything demonstrated the erratic movement. After a while, he said "To mention all the mineral substances in which I have found these molecules, would be tedious." It may be tedious, but he basically did mention all of them! Thus, he concluded that the erratic movement is not due to some life force.

THE SECOND BROWN ARTICLE

Brown actually wrote two articles concerning Brownian motion. The second one [11], called "Additional Remarks on Active Molecules," is written "to explain and modify a few of its statements, to adver to some of the remarks that have been made . . . ."

Mostly he wanted to make sure that ". . . an erroneous assertion of more than one writer, namely, that I have stated the active molecules to be animated" does not go down in history; he certainly did not want to go down in history as someone who thought they were alive! He also made clear that his experiments aimed to show his "belief that these motions of the particles neither arose from currents in the fluid containing them nor depend on the intestine motion which may be supposed to accompany its evaporation." He goes on to make clear that we do not know the cause, and he hopes future experiments will "ascertain the real cause of the motions in question."

[OF THE GREAT SCIENTIFIC DEBATES, THE EXISTENCE OF ATOMS RANK AMONG THE VERY TOP.]

WHO DISCOVERED BROWNIAN MOTION?

At the end of his second article, Brown clearly wanted to correct his lack of proper review of the literature: "I shall conclude these supplementary remarks to my former observations, by noticing the degree in which I consider those observations to have been anticipated." He went on to mention that it seems Leeuwenhoek, the inventor of the microscope, probably saw Brownian motion. He also named many scientists who studied Brownian motion and gave a brief account of his predecessors' actions. The names are Gray, Needham, Buffon, Gleichen, Wrisbur, Muller, Drummond, Bywater, all are essentially forgotten, so we might as well mention them here.

After Brown, the possible causes of this erratic movement were studied. Many possibilities were suggested, including vibrations, temperature fluctuations, light, surface tension effects, and electricity. It was 1880 when Georges Gouy, a French physicist, saw everything clearly. He was an experimentalist and did careful experiments that could leave no doubt that the erratic motion could not be due to external effects. He argued that it was some inherent property of the fluid that was causing the motion, and he proposed reasons that were very close to the truth. In fact, he argued that it was a direct reflection of atoms. Thus, he clearly had the idea before Einstein, but he did not work out the concrete results that Einstein did.

THE ATOM AND RANDOMNESS: A QUICK HISTORY TO 1908

Of the great scientific debates, evolution, the age of the earth, the central position of the sun, the existence of atoms rank among the very top not only because of the central question of their existence, but because of the peripheral issues, among them, the frightening thought that thermodynamics, the science dealing with the most fundamental principles, is *mere* statistics. The history of the atom starts slowly and builds into a story of the greatest magnitude until the early part of the 20th century when it is resolved, although nobody "saw" an atom until many years later. Atoms, historically speaking, started structureless but turned out to be incredibly structured objects with a beauty and depth that no one could have imagined. That these atoms can combine to form molecules, creating substances whose properties are so dramatically different than the original atoms; that we can combine gases to form water; poisons to form salt; and that there seems to be no end to the variety of molecules, is certainly something that no one could have dreamed about.

The possibility that matter, clearly perceived by our senses as continuous, is really composed of discrete objects began to enter serious thought around the early 18th century. The originator was Bernoulli. But wait, what about the standard line that Democritus, the laughing philosopher, conceived of the idea of atoms? Take his ideas for whatever you want, but it was not science. Democritus explained nothing with it, and it was one idea among numerous ones that turned out to have some semblance to reality. We forget all the other nutty ones arrived at by pure speculation. One of Democritus's ideas was that atoms are indivisible. So much for speculation. Aristotle made fun of the idea, and it was forgotten until someone thousands of years later remembered him. In the words of Jeans [12], "Given that a great number of thinkers are speculating as to the structure of matter, it is only in accordance with the laws of probability that some of them should arrive fairly near the truth."

In 1738, Bernoulli did have a scientific reason for inventing atoms and did explain something with them. Boyle, some 70 years earlier, had shown that air exerts pressure and that it is inversely related to volume. So the question became: How come? How does air exert pressure? Boyle himself came up with the explanation that, somehow or other, particles repel each other; Newton and others took up this idea. Newton developed the "repulsion theory" of a gas and combined it with the then-prevailing idea that heat is a fluid. Bernoulli, on the other hand, saw clearly that if the gas consists of little balls, then pressure arose from the force with which they hit the sides; he derived Boyle's law using this idea. That is the derivation we now see in elementary physics and chemistry books. But no one took any of this too seriously; if anyone did worry about atoms, it was with the Boyle-Newton idea, not with Bernoulli's idea.

LAVOISIER, DALTON, AND AVOGADRO

Nothing much happens until the end of the 18th century and the beginning of the 19th century. Lavoisier, lawyer turned scientist, revolutionized almost all previous thought about the nature of matter. Forget all that fluff of the past 2,500 years—the earth, fire, air, water, view of nature—he said. Forget about earth, fire, air, and water as the four fundamental, indivisible elements of which everything is composed. He came up with the role of oxygen in combustion and explained what fire really is, and, as to the idea about the fundamental indivisibility of water, he decomposed it! He came up with conservation of mass in chemical reactions and numerous other ideas that are now standard and which caused a revolution in human thinking. But he did not pay attention to the real revolution, and had his head cut off because he once had the job of collecting taxes.

One of Lavoisier's great achievements was to understand that when substances react to form new substances, they do so in given mass proportions. This was the start of modern chemistry and, in particular, stoichiometry. But why should things always react in the same way? That question is what led Dalton, in 1803, to propose "atoms." He understood, in rough terms, that if there are atoms and these atoms react to form new substances that are fixed combinations of atoms, then substances must react in given proportions. At about the same time, Gay-Lussac came up with studies of how things combine when the volumes of the reacting gases are considered. Facts were catalogued but not understood. Enter Avogadro. In 1811, he came up with the key dramatic idea that could clarify everything. He argued that everything becomes transparent if we just assume that, in a fixed volume, the *number* of little balls of a gas is the same for all substances, no matter what the mass, the size, or the exact nature of the substance. Take a liter, fill it with any gas you want, and it will have the same number of balls, Avogadro said, and if you keep that in mind, then how and why things combine to form molecules becomes easily understood. This was the birth of Avogadro's number, except for the fact that he wasn't taken seriously, died lonely and unrecognized, and nobody named his number for many years. As we will see, noise helped restore him to his rightful place in history. This is crucial to the story because, indeed, Einstein and more particularly Perrin took it very seriously and passionately. In fact, Perrin is the one that coined it "Avogadro's number."

CLAUSIUS AND TEMPERATURE

Around 1850, Clausius was revolutionizing the concept of heat, and in particular, he was one of the leaders against the still prevailing theory that heat is a fluid that flows much like any other liquid. He is the one that formulated thermodynamics in the way we know it today. In addition, he was one of the originators of the kinetic interpretation of heat, that is the atomic interpretation of heat. In particular, he rediscovered Bernoulli's idea but also came up with the further incredible idea that temperature is a measure of the kinetic energy of each atom. So, now, we have a microscopic understanding of what temperature really is.

THE SMELL PROBLEM

It was about this time that very indirect methods were devised to estimate the speed of the molecules in gases. The speeds turned out to be spectacularly high. Well, if molecules really move so fast, then we should smell odors almost instantaneously. But we know that when someone starts cooking at one end of a room, it takes time for someone across the room to smell it. If these molecules moved so incredibly fast, why don't we smell odors almost instantaneously? Yes, molecules do move very fast, but they are hindered in their forward progress because they collide. Now, we suddenly had collisions, and this added a totally new dimension to the reality, or unreality, depending on which side you were on, of atoms. Also, Clausius asked for the average time that a molecule goes before colliding with another; thus the concept of mean free path was born. This perhaps is the first significant stochastic quantity in history. But all this had a further air of unreality. It gets worse.

MAXWELL: THE DISTRIBUTION OF VELOCITIES

Clausius assumed that at a given temperature, the little balls are all moving at the same speed. Maxwell, young but already very famous, made a major contribution to the existence of atoms. In a simple fashion, he argued and showed that the constant velocity assumption is not the case. He came up with the idea that at a given temperature, molecules are moving with all speeds, zero to infinity, but the fractional number at each velocity is distributed according to a Gaussian distribution. Of course, we now call that the Maxwell or Maxwell-Boltzmann distribution. Then Clausius argued that temperature is proportional to the *average* kinetic energy, which is the variance of the distribution. Now, temperature is stochastic! This was in 1860.

THREE REMARKABLE BOLTZMANN QUESTIONS

Boltzmann then became the leading atomist and also the focus of the anti-atomists. First, he asked: Why should the Maxwellian distribution be the distribution? Second, he asked: Suppose we have internal and external forces, then what should the distribution be? But the most profound question was: Since we can start a gas with any distribution of velocities, who can stop us after all, then how and why does it *evolve* to a Maxwellian?

THE BOLTZMANN EQUATION— THE BIRTH OF STOCHASTIC PROCESSES

Look at it another way. These atoms are going all over the place, moving like crazy in all kinds of ways with all different speeds. So how is it possible that these fantastically erratic movements evolve to a Maxwellian and stay Maxwellian? These questions and answers make Boltzmann one of the greatest scientists of all time. He realized that collisions are at the root of it and derived what is now one of the most famous equations of science, the Boltzmann equation. As to the approach to steady state or equilibrium, Boltzmann showed that his equation evolves to a Maxwellian. For those who do not know this equation, it must be stressed that it is one of the most important equations in physics, astronomy, chemistry, and plasma physics, and we could go on and on. It is an equation of evolution for the probability density of position and velocity, $f(\mathbf{r}, \mathbf{v}, t)$,

$$\frac{\partial f}{\partial t} + v \cdot \nabla_r f + F \cdot \nabla_v f = \left(\frac{\delta f}{\delta t}\right)_{\text{coll}}, \qquad (3)$$

where F is the external force. The right-hand side is the so-called collision term. It is the change in the distribution due to collisions. Boltzmann wrote it explicitly for a dilute gas, and it is standard now for that case. For other cases, there are different types of collision terms. Because of this term, the process is irreversible and indeed goes to Maxwellian; hence, the terminology Maxwell-Boltzmann distribution. It was Boltzmann who first fully understood the concepts of stochastic processes, particularly the idea of an evolving probability distribution. In his classic book *Lectures on Gas Theory*, one can find many of the ideas of stochastic processes as we now know them. This equation should be seen as the start of "stochastic processes" as a field.

ENTROPY: CLAUSIUS AGAIN AND BOLTZMANN AGAIN

Clausius enters again with a totally new idea—entropy. Thermodynamics had become a fundamental science, perhaps the most fundamental of all sciences, with laws that are so general and so powerful no one could doubt their absolute validity. It permeated all fields and was the foundation of the industrial revolution. The first law, the conservation of energy in its most general form and the equivalence of different forms of energy, was formulated by Helmholtz around 1850. Carnot came up with a means to study and formulate a cyclical thermodynamic process. However, there was clearly something missing from thermodynamics: sometimes things return to how they were, but most often they do not. Entropy, as a thermodynamic quantity, was thus invented by Clausius to formulate "irreversibility." The definition he gave was strictly thermodynamic, and it was soon established as a fundamental part of thermodynamics. Entropy never decreases in a closed system, it either remains the same or increases, and that is a measure of irreversibility. This was around 1865.

Of course, Boltzmann then asked: What is entropy?—from my point of view, my point of view being that I and a few others really know that the world consists of little things called atoms. What is entropy from that point of view? Boltzmann came up with the epitaph to his tombstone

$$S = k \log W, \tag{4}$$

where k is the Boltzmann constant and W is the number of states accessible. This was around 1872. The atomic idea started to get serious as an *idea*, but even as an idea it was very disturbing since it was hard to swallow that bedrock thermodynamics was *merely* statistics. Many just smiled at the idiocy of the thought. Also, let's not forget that nobody ever dreamed of seeing atoms or of having a concrete experience of their existence. No one could demonstrate a real manifestation of atoms that would turn the tide against the nonbelievers. And, in fact, there were simple and powerful arguments against atoms. However, we point out that the idea of atoms was more accepted by chemists, one of the reasons being that Dalton explained multiple proportions with the concept.

THE GREAT DEBATE

The existence of atoms became intertwined with Boltzmann and entropy. It makes sense that it should be so, and the arguments went something like this: If you take atoms seriously, derive equations, make thermodynamics mere statistics, and define entropy in terms of movement of unseen atoms, well then, if we find a fundamental flaw, then not only are all your mathematics and ideas wrong, but perhaps it will put an end to the whole silly atom idea. The first argument against Boltzmann was made by his friend Loschmidt, who actually believed in atoms, but was not shy in bringing up important counter arguments.

REVERSIBILITY ARGUMENT

Loschmidt's reversibility argument is simple and powerful. Newton's equations are time reversible. What that means is that if you solve them and get the positions and velocities of the particles at a future time, then, if you reverse their velocities, the particles will trace back to where they started. Therefore, if we have a system that evolves, and entropy is a function of the particle coordinates, we can easily create a system (by reversing the velocities) that obviously has decreasing entropy, since the system goes back step by step to where it came from. So entropy doesn't always increase, yet we know it always does according to thermodynamics, so there you have it! This argument was simple and powerful and, moreover, it was put forward by a friend! Boltzmann had no simple convincing answer. This became a centerpiece of the antistatistical interpretation of the nature of matter. Boltzmann once said of this argument: Try it. Boltzmann knew it was not a good rebuttal. This was around 1875. Also, we point out that Loschmidt made many contributions to the atomic view and indeed was one of the early workers measuring atomic properties.

THE RECURRENCE ARGUMENT

Poincaré was a reigning figure—an astronomer, physicist, mathematician, and philosopher of science. He proved a remarkable theorem. For certain systems of particles, as they evolve, the system will eventually come back to almost the same initial conditions. This is a remarkable theorem in classical mechanics and crucial to the understanding of the evolution of N particle systems. Zermelo, Poincarés's student, who was to become Boltzmann's enemy, argued: if you wait around, then any system will come back more or less to where it was and, hence, entropy as defined by Boltzmann will come back to more or less what it was. Therefore, entropy doesn't always increase according to the Boltzmann idea. Boltzmann shot back saying well, yes, but it would take a very long time, but, Zermelo answered, that's not the point! Does entropy never decrease, as thermodynamics says? Or does it indeed decrease according to you? Yes or no, don't hedge, he argued. Boltzmann had many replies, but not good ones for his time. We also mention that indeed it was Poincaré, in 1890, who really first made the argument, but it was Zermelo who pursued it.

BOLTZMANN

Boltzmann was a grand man in every way—in size, personality, appetite, travel, excitement, charm and, of course, accomplishments. Also, he suffered from depression, got mad as hell at times, attempted suicide, and succeeded. But, basically, he was a nice guy who felt he was revolutionizing science and did not understand why some were opposing him so viciously. Many, of course, were on his side, and some of them were the greatest scientists. Years later, Lisa Meitner, the discoverer of nuclear fission, would remember Boltzmann's lectures as "the most beautiful and stimulating that I have ever heard . . . He himself was so enthusiastic about everything he taught us that one left every lecture with the feeling that a completely new and wonderful world had been revealed."

THE HEAVYWEIGHT ANTI-ATOMISTS: OSTWALD AND MACH

There were two super heavyweights leading the anti-atomist view, Ostwald and Mach, and they were heavyweights of major proportions. Ostwald was considered the greatest chemist of his time. He made many contributions to all aspects of chemistry, particularly electrochemistry, and is generally credited for starting the field of physical chemistry. In addition, he was very influential in the sense that he started many major journals and had many great students who were also influential, a number of them Nobel Prize winners. So there you have it, the inventor of physical chemistry did not believe in atoms and was not exactly shy in expressing his views very strongly. Ostwald had an agenda for opposing atoms. He, like everyone else, realized that the fundamental nature of matter was the greatest problem of all time. He had his own theory, called energetics, and it didn't involve atoms at all. We point out, though, that Ostwald and Boltzmann were friends, more or less.

[BROWN DID NOT DISCOVER BROWNIAN MOTION, BUT HE STUDIED IT SERIOUSLY, SYSTEMATICALLY, EXHAUSTIVELY, AND PASSIONATELY.]

Mach, one of the great physicists, who was the inspirer of Einstein on many issues but particularly on what has come to be known as Mach's principle, argued: you can't see them and you don't need them. It's the super positivist view of nature. Mach also was not exactly shy and the debates raged on bitterly. To make the drama even more interesting, Ostwald and Mach did not really get along because Ostwald, seeing a great potential ally, wanted Mach to be for his energetics approach; Mach wouldn't bite, and thus Ostwald wasn't exactly happy.

Everybody got involved in the atomic debate—poets, philosophers, everybody. There were also many great scientists, like Planck, who opposed atoms and then changed their minds. Of course, anything so fundamental as dealing with "reality" had to attract everyone. I know of no better way to portray the flavor and intensity of the debate than to give some quotes from various time periods:

"I don't believe that atoms exist!"—Mach

"If I were the master, I would outlaw the word 'atom' from science . . . " —Dumas

"We shall never get people whose time is money to take much interest in atoms"—Samuel Butler

"I accept neither Avogadro's law, nor atoms, nor molecules"—Berthelot

"Every time someone has tried to imagine or depict atoms . . . in short a sterile conjecture"—Deville

"Atomism is a doctrine that has miserably failed . . . " —Ostwald

" . . . atoms . . . absolutely contradict the attributes hitherto observed in bodies"—Mach

POLITICALLY CORRECT ATOMS: READ THE BOOKS BACKWARDS

The debate had a long-range effect, and the nonbelievers made it more than just a great scientific debate. In his book [13], Pullman describes the effect of the governmental decrees in France not to teach atoms, which was due mostly to the political influence of Berthelot. As we can see from the quote above, Berthelot didn't like Avagadros's number, atoms, nor molecules, and he did something about it. To further quote Pullman, "Bertholet's actions had a disastrous consequence on the teaching of chemistry, on research, and even on industrial development." He also quotes the chemist Bachelard, who experienced it himself: "Most textbooks complied with peculiar government decrees by mentioning the atomic hypothesis as an afterthought at the very end of the chapter devoted to chemical laws. Worse, some relegated it to an appendix to emphasize that chemistry had to be thought in an untainted positivist form . . . . the trick was never to utter the word 'atom' . . . Alas, . . . these 'politically correct' books would have made more sense had it been permissible to read them backwards."

WHAT BECAME OF OSTWALD AND MACH?

After Einstein and Perrin, Ostwald, in 1908, changed his mind, and he was big about it. He received the Nobel Prize in 1909 and remained active until his death in 1932. He is remembered as one of the greatest scientists ever. The Nobel Prize was "in recognition of his work on catalysis and for his investigations into the fundamental principles governing chemical equilibria and rates of reaction." His Nobel lecture is a beautifully written history of many parts of chemistry and physics. It is, of course, ironic that everything he discussed, almost all of physical chemistry, is now explained in terms of atoms! Mach, on the other hand, never relented. He wrote a famous book on mechanics called the *Science of Mechanics,* which went through many editions. I recently read all the introductions to the various editions. No mention of atoms there, but he does mention them at the end of the book.

EINSTEIN: WHAT DID HE DO?

Remember, Einstein's aim was to find a measurable experimental manifestation of atoms, to predict a macroscopic observable fact. In a few pages, he introduced fundamental stochastic arguments that are common today. He derived the probability distribution for the small particle, obtained the spread to be proportional to the square root of time, and also described how one can measure Avogadro's number from a stochastic quantity!

For the rest of this section, we use Einstein's notation. He first derived the governing equation for the probability of the Brownian particles, $f(x, t)$, to be at position x at time t

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{5}$$

and pointed out that, of course, "this is the well-known equation for diffusion . . . D is the coefficient of diffusion." He solved it for the impulse response (delta function of position at time zero, normalized to n, the number of small particles) and gave

$$f(x,t) = \frac{n}{\sqrt{4\pi D}} \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{t}}. \quad (6)$$

Einstein then explicitly wrote the standard deviation

$$\lambda_x = \sqrt{\overline{x^2}} = \sqrt{2Dt}. \quad (7)$$

The formula is pretty clear, but he emphasized what will become the most famous fact of Brownian motion: "The mean displacement is therefore proportional to the square root of time." However, the truly great result is that Einstein had, a few paragraphs earlier, derived the diffusion coefficient in terms of the fundamental constants

$$D = \frac{RT}{N} \frac{1}{6\pi kP}, \quad (8)$$

where R is the usual gas constant, T is the temperature, k is the viscosity, and P is the radius of the small particle. (Note that I have kept totally to Einstein's notation. It is important to emphasize that k is now universally used for Boltzmann's constant, but in Einstein's 1905 paper he uses it for the viscosity.)

Here is the main point: combining (7) and (8), we obtain

$$\lambda_x = \sqrt{t}\sqrt{\frac{RT}{N}\frac{1}{3\pi kP}}, \quad (9)$$

all terms being measurable or known approximately. So either this equation is true or not, and Einstein stated the consequences of the equation's validity in his introduction. To make the point that one can actually measure λ_x, he took room temperature and got what λ_x would be per minute, and then arrived at an easily measurable number. As if that were not enough, Einstein then reversed the argument. Solve (9) for N (for time equal to one second, say)

$$N = \frac{1}{\lambda_x^2} \frac{RT}{3\pi kP} \quad (10)$$

and hence N can be measured this way. Let us not forget that N is Avogadro's number and hence fluctuations, noise, can be used to measure a deterministic number, one of the most important numbers in science.

THE KEY IDEAS

Besides the ideas mentioned, it is important to appreciate the following fact. Einstein saw that the heavy particle is just a big atom pushed around by the real atoms, and according to energy equipartition, the statistical properties of the big particle are the same as the real invisible atoms. More precisely, the mean kinetic energy of the pollen is the same as the mean kinetic energy of the atoms. Therefore, we can use the heavy particle as a probe of the ones we cannot see. Think of this analogy: an elephant is on ice and gets hit statistically by mosquitoes, but because of equipartition some of the statistical properties of the elephant are the same as that of mosquitoes. Hence, if we measure the statistical properties of the elephant, we will know the statistical properties of the mosquitoes. But, more importantly, we can conclude that mosquitoes exist by the erratic movement of the elephant. (See also "The Mathematics of Noise.")

Many books and articles imply that Einstein did the free particle case subjected to a random force and Langevin did the case with friction. That is not correct at all. Einstein did do it with friction, but he did it separately, and, in fact, that is how he related D to the other physical quantities mentioned above. Also, we point out that the cause of the fluctuations, the atoms, are also the cause of the friction because once the particle moves, it gets hit more from one side than the other and hence gets slowed down. This is why there is a fluctuation-dissipation theorem that relates the fluctuations to the dissipation.

WHY WAS EINSTEIN PREPARED TO DO IT?

Einstein was totally involved with atoms, their physics, and chemistry. It was one of his main interests, and his thesis and a number of papers dealt with estimating the mass of atoms. So he lived with the practical issues of atoms on a daily basis. Don't think of the Einstein of the newspapers and of unified field theory, but of Einstein as a down-to-earth practical chemist and physicist. To impress this upon the reader, in one of his papers on Brownian motion, for example, one finds "Diisoamyl-ammonium, $C_{10}H_{24}N$." I don't know what that is, but it's part of a practical numerical discussion he gives on measuring diffusion and viscosity. Hence, he did not just wake up one morning and come up with the idea of proving the existence of atoms. And, most importantly and contrary to what is often implied in many books on stochastic processes, he was not after explaining Brownian motion. He predicted Brownian motion as an observable phenomenon with the aim of proving the existence of atoms.

LANGEVIN, THE EQUATION, PICASSO, AND STOCHASTIC DIFFERENTIAL EQUATIONS

Paul Langevin was a physicist who made many important basic discoveries in a number of fields, particularly on the magnetic properties of materials. He was extremely productive and famous in his lifetime, and he is now considered one of the great physicists of the last century. In his later years, Langevin worked on sonar, the associated electronics, and many other things. His four-page paper is usually taken as the birth of stochastic differential equations [14].

Langevin was a student of Pierre Curie. Langevin and Marie Curie caused a major scandal that was not his fault or Marie's, but that of France! Langevin and Curie had a love affair about five years after Pierre Curie died, but Langevin was married and his wife wasn't exactly thrilled. It made the newspapers big time, and a scandal ensued. At times, the public protests almost turned violent. The reason behind the scandal and the mobs was that a foreigner was corrupting a French scientist! After all, Marie was born in Poland! The cries were to run her out of town, that is, out of France! Langevin stuck by her in every way and even had a duel with a reporter. Of course! At one point, there was a mob in front of her house, and the great mathematician Borel, who at that time was an administrator at the Ecole Normale Supérieure, took her and her two daughters into his house for safety. He was immediately threatened by a prominent minister to give her up to the mobs. To his honor, he resisted. All this happened after she had won one Nobel Prize and during the time it was announced she would receive another. In fact, the Nobel committee told her to stay home, they couldn't take the heat, they said. But Marie wanted to see Sweden again and went there anyway. Today, France loves everybody; Paul and Marie are national heroes. They are on stamps, and streets are named after them. By the way, only a little while before this scandal, Marie was involved in another one; she had the nerve to run for membership in the French Academy of Sciences. She lost. What nerve! A woman! With only one Nobel Prize! And a Pole, no less. She was even accused of being Jewish! She wasn't. Incidentally, we mention for the sake of believers in cosmic love that many years later, Langevin's grandson married Marie's granddaughter.

LANGEVIN AND PICASSO

Picasso sketched Langevin. Why would he do that? It's one of the few, perhaps only, triumphs of communism. In Picasso's own words: "While I wait for the time when Spain can take me back again, the French Communist Party is a fatherland to me. In it I find again all my friends—the great scientist, Paul Langevin." This was in 1945.

WHAT IS THE LANGEVIN EQUATION AND WHAT DID HE REALLY DO?

The main aim of Langevin's classical 1908 paper, which is now taken as the birth of stochastic differential equations, was to derive Einstein's result in a more physically transparent way that also simplifies the mathematics. This view regarding simplification of the mathematics is not quite correct, as we will shortly discuss. The title of the paper is "On the Theory of Brownian Motion." It is a very short article, four pages. The aim was to derive the Einstein equation (9), that is, to obtain an explicit expression for $\overline{x^2}$, from which the standard deviation follows ($\overline{\Delta_x^2}$ in Langevin's notation, λ_x in Einstein's). It is interesting that he gives full credit to Gouy for the basic idea that Brownian motion is a "manifestation of molecular thermal agitation." However, he immediately says that "quantitative verification . . . has been rendered possible by A. Einstein." He also very appropriately gives credit to Smoluchowski: "has tackled the problem . . . he has obtained for $\overline{\Delta_x^2}$ an expression of the same form [as Einstein, (7) here] but differing by the factor 64/27."

Langevin's approach was to use Newton's equation, which is direct if you know the forces. It was known that for a ball going through a fluid, there is a friction force proportional to the velocity, which of course also depends on the size of the ball and on the viscosity. This force is called Stokes law, and it appears in every elementary physics textbook then and now. It is given by $6\pi\mu a\xi$, where ξ (using his notation) is the velocity, a is the radius, and μ the viscosity. Langevin argued that, in addition to friction, there is another force because "in reality . . ." the Stokes force "is only an average, and because of the irregularity of the collisions of the surrounding molecules . . ." The total force is, hence, the friction force plus X, the irregular force. Writing $ma = f$, we have

$$m\frac{d^2x}{dt^2} = -6\pi\mu a\frac{dx}{dt} + X, \tag{11}$$

where "X is indifferently positive and negative." Of course, X is what we now call the random force. This was the first time in history that Newton's equation was used with a random force. The negative sign in front of the friction force is because it always opposes the motion.

The whole aim is to get $\overline{x^2}$, and so Langevin had to get x^2 in (11), which he did by simply multiplying by x to get

$$\frac{m}{2}\frac{d^2x^2}{dt^2} - m\left(\frac{dx}{dt}\right)^2 = -3\pi\mu a\frac{dx^2}{dt} + xX. \tag{12}$$

(We have changed his notation slightly by using dx/dt instead of ξ, which indeed he does himself subsequently). He then took the average of both sides and said that the average of xX is "evidently null by reason of the irregularity of . . . X" to get

$$\frac{m}{2}\frac{d^2\overline{x^2}}{dt^2} - m\overline{\left(\frac{dx}{dt}\right)^2} = -3\pi\mu a\frac{d\overline{x^2}}{dt}. \tag{13}$$

To eliminate $\overline{(dx/dt)^2}$ he used

$$m\overline{\left(\frac{dx}{dt}\right)^2} = \frac{RT}{N}, \tag{14}$$

which relates it to the macroscopic temperature, T. This substitution is crucial and, of course, was also used by Einstein. The reason we can equate the two is because of equipartition of energy, which stems from the fundamental idea that the pollen grain is just acting like an atom but with different mass (see below). In (14), N is "the number of molecules in one gram-molecule, a number well known today and around . . . " That is, the yet unnamed Avogadro's number for which Perrin would get the Nobel Prize years later. So here Langevin used an estimated value, while Einstein used the reverse argument, and said we can measure N.

The solution of (13) is

$$\frac{d\overline{x^2}}{dt} = \frac{RT}{N}\frac{1}{3\pi\mu a} + Ce^{-\frac{6\pi\mu a}{m}t} \tag{15}$$

and he argues that, for the standard values of the parameters, the second term is very small and hence

$$\frac{d\overline{x^2}}{dt} = \frac{RT}{N}\frac{1}{3\pi\mu a} \tag{16}$$

or

$$\overline{\Delta x^2} = \frac{RT}{N}\frac{1}{3\pi\mu a}\tau, \tag{17}$$

which is identical to Einstein's result, (9). (Note here the viscosity is μ, but in Einstein's paper it is k).

We now crystallize why Langevin's contribution is so important. Most prominent is the idea that we apply Newton's equation directly as if it was a deterministic problem and then worry about the stochastic issues after simplification or after writing the solution for the deterministic equation. This will become the standard method used for stochastic differential equations. (See "The Mathematics of Noise.") To introduce the stochastic part, Langevin ensemble averages, which brings in the stochastic properties of the random force.

WHY IS THIS SIMPLER THAN THE EINSTEIN PROCEDURE?

Langevin's approach is simpler because it is easier to write Newton's equations once one knows the forces. However, the Langevin procedure doesn't get you as much, but it does give you the important quantities directly and easily. The Einstein procedure gives you the whole distribution while the Langevin procedure gives only the first and second moment. The Langevin method avoids writing an equation for the distribution of x. Of course, it is often very important to get the probability distribution; the methods to do so are now called Fokker-Planck equations or Master equations. Hence, if one just wants a few low-order quantities like the mean and standard deviation as a function of time, then the Langevin approach is indeed much easier. If one wants the distribution of x, then it is not. Of course, one can in principle calculate all the moments from the stochastic differential equation and then write the distribution, but that is generally not feasible except for simple cases. For further discussion on this point see the section titled "The Mathematics of Noise."

WHAT IS EQUIPARTITION OF ENERGY?

This concept of equipartition of energy is crucial to discussions of noise, and perhaps this is a good place to explain it. Think of a big box containing a lot of elephants, a lot of mosquitoes, and a lot of molecules, all of them colliding with each other. Forget about gravity. Equipartition of energy is a remarkable result that shows that, in equilibrium, the mean kinetic energy of each species is the same! That is

$$\begin{aligned}\frac{1}{2}m\left\langle v^2\right\rangle(\text{of elephants}) &= \frac{1}{2}m\left\langle v^2\right\rangle(\text{of mosquitos})\\ &= \frac{1}{2}m\left\langle v^2\right\rangle(\text{of molecules}).\end{aligned} \tag{18}$$

We can see elephants, but we can't see molecules. And, if we are far away, we can't see the mosquitoes. But the motion of the elephants gives us a window to the mosquitoes and molecules. Now $\langle v^2\rangle$ of the elephant is very small because its mass is very big, but if we could see and measure it then we can tell that there are small things affecting it (in this case mosquitoes and molecules). This is the main reason why we can use Brownian particles to prove the existence of molecules. The Brownian particles are the elephants. Furthermore, the kinetic energy is proportional to temperature and hence the "temperature" of the elephants and molecules and mosquitoes are the same

$$T_{\text{elephants}} = T_{\text{mosquito}} = T_{\text{molecules}}. \tag{19}$$

In particular for three-dimensional motion

$$\frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT = \frac{3}{2}\frac{R}{N}T, \tag{20}$$

where k is Boltzmann's constant and equal to R/N. Therefore, one can relate the microscopic quantity $m\langle v^2\rangle$ to the macroscopic temperature, T. Everybody has the same temperature and, hence, everybody has the same mass-velocity combination given by $1/2m\langle v^2\rangle$.

PERRIN

It is often written that Perrin verified Einstein, hence establishing that atoms exist, got the Nobel Prize, and the rest is history. That is not quite correct. Perrin was devoted to atoms and to the measurement of Avogadro's number prior to Einstein's paper. As previously mentioned, he was the one who coined "Avogadro's number." We mention again that Avogadro had no idea of the magnitude of his number but saw clearly that if reactions take place the way they do, then the number of atoms in a given gas volume must be the same, independent of the substance. A preposterous idea on the face of it, but true. Perrin actually considered many ways of measuring Avogadro's number. Of course, Einstein's 1905 article motivated him even more and, in addition, things were really heating up in the atomist debate. One of the ways Perrin considered was based on Einstein's formula. In fact, Perrin used a suspension, and in that case one must take into account the external force of gravity. This was done not by Einstein but by Smoluchowski.

Perrin published a very beautiful and influential book in 1913 called *The Atoms* [15]. It is a short but sweeping book that discusses many topics, including the theory of density fluctuations. In the book and in other places, Perrin discussed many different ways of measuring Avogadro's number and argued that since these different ways give roughly the same results, there must be a reality to all of this! He got the Nobel Prize in 1926, and his Nobel lecture makes fascinating reading. Even though it was written in 1926, after everyone had accepted the idea of atoms, he never lets the reader forget the central issue. He starts with: "Since I have the great honour to have to summarize here the work which has enabled me to receive the high international distinction awarded by the Swedish Academy of Sciences, I shall speak of the 'discontinuous structure of matter' . . . ." He then gives a beautifully written historical blow-by-blow account of atoms. He goes into detail, but every few paragraphs he reminds the reader about the main issue. At one point Perrin says: "Indeed, increasingly numerous and strong reasons have come to support a growing probability, and it can finally be said the certainty, in favor of the hypothesis of the atomists."

AVOGADRO'S NUMBER AND COUNTING ATOMS

In the words of Poincaré (about 1910): "the atomic hypothesis has recently acquired enough credence to cease being a mere hypothesis. Atoms are no longer just a useful fiction; we can rightfully claim to see them, since we can actually count them." Perrin counted them.

BACHELIER, THE STOCK MARKET, AND FINANCIAL ENGINEERING

The field of mathematical finance, sometimes called financial engineering, considers Bachelier as its father. That is quite proper and justified. Moreover, it can be said that Bachelier should be recognized as one of the originators of stochastic processes. Anyone who is not familiar with current mathematical economics would be shocked to know the depth of the study of stochastic processes.

While Bachelier's famous thesis addresses the issue of stock prices, it must be emphasized that his lifelong interest was in probability and stochastic processes. His approach in the thesis was as a scientist, and he wanted to see how stock prices behaved in time from a probabilistic point of view. (Actually, he was studying option prices, which have always been the most interesting time series because they have a finite life time, among other reasons.)

Bachelier took courses given by Poincaré who, as previously mentioned, was one of the greatest mathematicians, physicists, astronomers, and philosophers of science. In astronomy, Poincaré is a legend because of his work on dynamics. Of course, he is the discoverer of the sensitivity of initial conditions in dynamics, a crucial part of what we call chaos theory today.

Bachelier's thesis [16], "Theory of Speculation," was completed in 1900. In the thesis, he developed many mathematical ideas, that are now part of stochastic processes. Of course, the fluctuation of stock prices was an unusual thing to take up. Poincaré's summary of the thesis defense clearly shows that everyone was aware of the unusual topic, but it is also clear that everyone was impressed with the results, particularly the mathematics. His thesis committee was as impressive as any in history. Besides Poincaré, there was the physicist Boussinesq, who was a major contributor to almost all branches of physics but particularly hydrodynamics and turbulence. There was Appell, one of the great mathematicians of his time. I emphasize this for a reason that will be clear shortly. The thesis report written by the committee emphasizes the main mathematical results and is very positive. "The manner in which M. Bachelier deduces Gauss's law is very original and all the more interesting in that his reasoning can be extended with a few changes to the theory of errors."

Of the many significant results Bachelier came up with is the idea of the transitive property of a probability distribution,

$$P_{z,t_1+t_2} = \int_{-\infty}^{\infty} P_{x,t_1} P_{z-x,t_2} dx, \qquad (21)$$

and shows that the solution is given by (6), derived by Einstein five years later.

We now describe what has become a famous story in regard to Bachelier's life. He had difficulties as a young person for a variety of reasons, including the loss of his father at a young age and the need for him to help out in the family business. However, he clearly did fine given that he was involved with Poincaré; that speaks miles. Bachelier was fully aware that his contributions were important and sometimes considered himself, perhaps arrogantly, as a major figure. He published quite a few papers and books on probability and had a number of positions, but most of these were temporary and minor. In 1926, he applied to the University in Dijon for a chair position. Bachelier was by now 56 years old. As was common at that time, a formal report had to be written, and it was assigned to M. Gevrey, a mathematician at Dijon who knew nothing about probability theory. He consulted Paul Lévy, who was and is considered a major mathematician and who made important contributions in the field of probability. Together, Gevrey and Lévy wrote a killer report because they said they found a major mistake! A killer mistake. No job for Bachelier and, moreover, this ruined his reputation. Bachelier got mad, really mad. And rightfully so. He said "The critique of Lévy is simply ridiculous." Bachelier explained why it was ridiculous and accused Lévy of, let us say, not being particularly honorable. Of course, this was to no avail, and Bachelier suffered the rest of his life. Much has been written about this affair, and the general view is that Lévy was honest in his assessment, although very mistaken in having found a "mistake." Also, the story goes that many years later, they made up and everybody kissed each other. The affair and ending do not smell right. How is it possible that someone like Lévy would think he found a mistake knowing full well that Bachelier was not exactly an amateur and was a student of the great Poincaré? How is it possible that if he really did think he found a mistake, he wouldn't first check himself out with others before using this mistake to thrust the sword? How is it possible that someone of Lévy's stature and accomplishments could be so cavalier, so irresponsible, and so silly? I have seen more letters of recommendation for tenure and promotion and referee reports than I have hair, and I am not bald. It has always shocked me what people are willing to say so cavalierly in a report often written in a few minutes and with the full knowledge that it may ruin people. I have been fascinated by this, and I believe that the source of this "courage" is self aggrandizement, cheap machismo, fear, and a few other similar things. I have read enough killer reports to suspect that, for some, looking in the mirror in the morning is a ritual: Mirror mirror on the wall, who is the best scientist of them all? and miraculously the mirror reveals the answer and instructs them to prove to the world that they are great, and what better way to do that than to dash off a killer report?

Bachelier was truly great, and it is very sad that he wasn't given his full due during his lifetime. As time passes, he will overshadow by far the poor treatment he got and will be remembered as one of the originators of stochastic precesses.

ECONOMICS

Many economists convey the notion that Bachelier was "discovered" by American economists after a life of obscurity. There is a PBS program that implies this romantic notion. This view is not correct and probably is just a case of PBS gone Hollywood. Bachelier was pretty well known by the greats of his time, such as Kolmogorov. It is also clear that Feller knew of Bachelier and thought highly of his contribution. Indeed, on page 181 of his classic volume 2, Feller equates "Brownian motion" and the "Wiener-Bachelier process." Economists should say that they became aware of Bachelier at a late date. It is proper to call him the originator of financial engineering or mathematical finance because, indeed, he was the first person to study econometric series in a serious manner. Incidentally, while we should be historically pleased that Feller mentioned Bachelier, it is curious he brought in Wiener but not Einstein, Smoluchowski, and Langevin.

[NOISE HAD A GLORIOUS BIRTH.]

THE VACUUM TUBE, NOISE, AND ELECTRICAL ENGINEERING

The vacuum tube changed the world. It was the device that initiated the study of noise in electrical engineering, and it is electrical engineering that has carried noise to the very highest levels in both theory and practice.

The 20th century was the electricity century. It brought the understanding of electricity and the beginning of its use in every way. For the man on the street, electricity went from being a parlor thrill for the rich to the main technology of everyday life. (It's fun to get shocked after all. When I was a teenager I used to fix radios, which was a common thing to do at that time. I decided to do it seriously and took an evening course for learning the trade. It was a "man" thing. One proved one's manhood by getting shocked straight from the wall. Some of us would cheat by using fingers on the same hand, but the real men would take it using two hands so that the shock would be felt through the chest.) The understanding of electricity and magnetism was seen as the greatest challenge since Newton, perhaps the greatest intellectual challenge of all time. Maxwell did it with what we now call Maxwell's equations, one the highest intellectual achievements of all time. Maxwell decided to use quaternions to express his equations, an unwise idea. Heaviside, using the new methods of Gibbs, namely vector analysis, re-expressed Maxwell's equations in a much simpler form, and that is how we know them today.

The electrical engineers of that time did not want to hear about Maxwell's equations. On one of the biggest projects of all time (in terms of normalized money), the building of the transatlantic cable, the engineers refused to take the new electromagnetism into account, resulting in many disasters that were solved by the Maxwellians. "Maxwellians" is a term used by Bruce Hunt in one of the best books ever written about science called *The Maxwellians*. Maxwell died early, and a group arose that pushed for Maxwell's equations in an extraordinary way. Among them were Heaviside, Hertz, Pointing, FitzGerald, Larmor, and Lodge. Using Maxwells' equations, they solved problem after problem that others could not.

Hertz, of course, achieved the most remarkable of successes. Standing on one side of a room, he produced a spark. On the other side, he had a "receiver" that consisted of a wire bent into a circle with a small gap. When he produced a spark on one side, he got a spark in the receiver, thus verifying Maxwell's prediction that there must be electromagnetic waves. Also, Maxwell's equations have a very strange property, in that they are not invariant to what Newton said all laws should be invariant to: observers moving with constant velocity. This property would lead Lorentz, Fitzgerald, and Einstein to change our view of the universe. In addition, it was Maxwell that produced the startling idea that ordinary light is an electromagnetic wave. Who could have guessed that the picking up of bits of paper by rubbing something on one's hair and the picking up of iron filling by some rocks would be connected to light?!

The technological uses of electricity came at a furious pace. Edison invented the stock ticker and didn't stop! The electric light bulb changed the world, as did most of Edison's subsequent inventions. It was Edison who discovered that, when a metal is heated, it gives off "electricity." That is, take the ordinary light bulb with a very hot filament and put a positively charged piece of metal in it, the plate, and current will flow in the light bulb from the hot filament to the plate, even though there is no physical connection between them.

Electricity in 1900 was the future technology and was seen to be as big as the mechanical technology of the previous 100 years. The inventions came out of necessity. The electric light bulb required the invention of generators. Edison invented them, as did Tesla, who made a deal with Westinghouse. Edison's was dc and Westinghouse's was ac; the ac version was clearly better. New York was the first big city to be electrified, and the cost involved was astronomical. Financial fights of all kinds broke out. As for the generator fight, the issue became: Do we choose the ac of Westinghouse or the dc of Edison? The electric chair was invented, and Sing Sing prison wanted to acquire both an electric chair and a generator. Of course, no one wanted their generator to be associated with death. The prison bought an ac generator and used it for the first electrocution. The dc guys then said, see what ac can do to you? It can kill you! (So for this and a few other reasons, I could not have a television in the

1950s because I lived in a building that had dc, as was the case in many New York buildings. But the force of TV was overwhelming, and buildings were converted swiftly. But even in the 1960s, many universities still had dc and students would buy dc-to-ac converters to watch their favorite cultural programs. The inventor of television, Vladimir Zworykin, said "I would never let my own children watch it.")

Of course there was Marconi and wireless, the telephone, and the transatlantic cable, among many other inventions that were only in their infancy. Then came the vacuum tube, which changed the world in almost every way, technologically and sociologically. It made worldwide communication possible, and it made the radio possible! If one were to go to the site of the World Trade Center before it was built, one would see numerous stores selling what seemed to be an infinite variety of vacuum tubes, some neatly shelved, others in barrels, priced anywhere from two to five cents. Corner drug stores had self-service machines to test vacuum tubes. It was a ritual when a radio malfunctioned to take out all the tubes and spend a half hour testing them to find the bad one. The names of the common tubes were known to everyone, even though most individuals had not the slightest idea how they worked. Electronics became one of the main hobbies of kids and adults.

The vacuum tube started with Fleming who, in 1904, invented the vacuum diode, known also as the kenotron, thermionic tube, or simply the Fleming valve. Its original function was to convert ac into dc, that is, to rectify. As mentioned previously, Edison discovered that current flows from the hot filament to a positively charged plate placed in a light bulb. This is because when something is really hot, electrons are given off and they go to the positive plate (if one is present) because the electrons are negatively charged. Edison did not do anything with the discovery, but the effect is fundamental; it is now sometimes called the thermionic emission or the Edison effect. But it was De Forest in 1905 who made the vacuum tube the great device it became. He added a "grid" in between the filament and the plate. The grid is just an ordinary piece of wire mesh, like the type used to keep mosquitoes out. By charging the grid differently, we can control the flow from the filament to the plate. Charge it very negatively and electrons will be repelled; charge it positively and electrons will be more than encouraged to flow to the plate. Therefore, the grid can control the rush of electrons. If we have small variations in the grid voltage, they will be amplified as large variations in the gushing current. Thus, amplification was born and so was modern communication since the weak signal from a transmitter miles away could now be amplified. The vacuum tube became the most important device until it was replaced by the transistor in the early 1950s. There was an incredible variety of vacuum tubes, and there were vacuum tubes with more than just three elements. A one-grid tube was called a triode because it had three elements: filament, grid, and plate. Tetrodes and pentodes were four- and five-element tubes. Catalogs of these tubes were published, each page describing the tubes' characteristics.

THE VACUUM TUBE AND NOISE

Everyone knew that the vacuum tube would play a prominent role in the future promise of electricity. It was as high tech as you could get. Noise played a fundamental role in the understanding and technology of the vacuum tube and, subsequently, in the technology of semiconductor devices. The main initial contributors were Schottky, Johnson, and Nyquist. Perhaps it is appropriate here to mention two other fields in which electrical engineering has produced great ideas in theory, mathematics, and engineering. Each of these certainly deserves a book of its own, but we only mention them briefly here. The first is modulation theory. In the development of radio, two methods of transmission were invented, amplitude modulation and frequency modulation. The behavior of additive noise on these two methods was a very important development, and much of noise theory stems from attempts to understand which method is less sensitive to noise. Also, modern communication theory is based on stochastic processes for many reasons. Among them is that the fundamental idea of transmitting information involves probabilistic considerations. This was initiated by the classic papers of Shannon.

SHOT AND THERMAL NOISE: SCHOTTKY, JOHNSON, AND NYQUIST

The common view that Schottky discovered shot noise and Johnson discovered thermal noise is not correct. Walter Schottky was a physicist who interacted with some of the great scientists of his time, such as Planck. Schottky's range of contributions is extraordinary, both in theory and experiment, and many effects carry his name. We have Schottky diodes, the Shottky barrier, the Shottky effect, among others. Also, he was perhaps one of the first scientists to be involved with both academia and industry, but he spent most of his life at the industrial laboratory, Siemens. He was one of the founders of semiconductor physics and solid-state physics. He was born in 1886 in Switzerland, but spent most of his life in Germany. Passing away at the age of 90, Schottky lived to see what he had wrought.

Schottky wrote his thesis on special relativity but was equally facile in vacuum tubes. He realized that the vacuum tube was the device that would revolutionize the world. He not only made fundamental contributions to vacuum tubes but to many other fields as well. Schottky, in a milestone paper in 1918, was the first to consider what is now called shot noise and thermal noise [17]. We emphasize that he considered *both*. What is now called shot noise is called such not because it is named after him. Schottky called it "shroteffekt," which in German means shot effect, with shot meaning pellets, like gunshot; hence, "shot" noise.

Think of the current flowing from the hot filament to the plate as being composed of baseballs; the current is proportional to the number of balls coming on average per unit time. Since the baseballs are being thrown at random, in the sense that the actual time of emission is not fixed but the average number is, the current has instantaneous fluctuations about the average, sometimes more and sometimes less than the average, with the variations adding up to zero. These variations are the "noise," in particular, the shot noise. The analogy often used is that of rain drops falling on a roof. During a steady downpour, the amount of water coming down is the same over the time scale of minutes. But imagine slicing time up in fractions of seconds. Then, the number of raindrops varies from one small time interval to another small time interval, and the difference produces a nonsteady sound. Compare, for example, the sound of raindrops on a roof with that of gushing water. It is the fluctuations that we call noise. Now, the common usage of "shot noise" is any stochastic process that is a sum of discrete events, the shot noise being the fluctuations around the mean value of the process. Incidentally, the existence of shot noise in a vacuum tube is evidence that electricity consists of little balls as, of course, it does, namely, electrons.

[IT WOULD BE A DULL, GRAY WORLD WITHOUT NOISE.]

Thermal noise is due to the fact that some electrons in a conductor are loose. In fact, they are so loose one can think of a piece of conductor as a box with electrons in it, just like the gas molecules in a room. The electrons are moving around randomly due to the same reason atoms move around randomly in a gas. On average, there are as many moving to the right as to the left. Hence, on average, the current, or the net charge moving, is zero if there is no applied voltage. However, there could be fluctuations, that is, at an instance there could be more electrons moving to the right than to the left, and the fluctuations produce a momentary current. The electrons have a distribution of speeds, and it is reasonable to assume that the wider the distribution of speeds, the higher the fluctuations. However, the width of the distribution is proportional to temperature and, hence, the effect is temperature dependent; thus, the phrase thermal noise.

Johnson was involved in the whole issue of noise from the beginning, and he clarified many issues in regard to both shot noise and thermal noise. According to Johnson [18], Schottky's paper "did not get to the United States until about 1920." Johnson was an experimentalist, but he was well versed in the theory involved. Many experiments were conducted, including those created in Schottky's laboratory, on these noise effects. While Johnson was involved in all the issues, over a period of years he did fundamental experiments on what we now call thermal noise. Both he and Harry Nyquist were at Bell Laboratories. They published back-to-back papers on the effect in *Physical Review* in 1928 [19], [20]. Johnson reported the experimental results and Nyquist gave a very general and very elegant derivation of the effect which, incidentally, led to many quantum mechanical discussions. Johnson's paper was 16 pages, and Nyquist's was just four. Johnson's abstract starts with "Statistical fluctuations of electric charge exists in all conductors, producing variations of potential between the ends of the conductor." In the introduction, he emphasizes that thermal noise "is often by far the larger part of 'noise' of a good tube amplifier" and then goes on to give the details of the experimental results he obtained. Nyquist's abstract is just two lines: "The electromotive force due to thermal agitation in conductors is calculated by means of principles of thermodynamics and statistical mechanics. The results obtained agree with the results obtained experimentally." The main idea is to relate the fluctuations to the temperature. He obtained the now famous result (in his notation)

$$E^2 d\nu = 4RkTd\nu, \tag{22}$$

where $E^2 d\nu$ is the square of the voltage in the frequency interval $d\nu$, and R is the resistance. Nyquist also commented that if we use Planck's law, we would get (I use angular frequency here)

$$E^2 = \frac{2R}{\pi}\left(\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}\right). \tag{23}$$

Nyquist comments that "Within the ranges of frequency and temperature where experimental information is available this expression is indistinguishable from . . . " (22). What he meant is that (23) reduces to (22) when $\hbar\omega \ll kT$, which was the case for the situations at that time. However, this will turn out to be a very important issue with quantum noise. (See the section "Quantum Mechanics, Quantum Noise, and the Laser.")

Of course, both Johnson and Nyquist are great figures in electrical engineering. Nyquist made many fundamental contributions, and certainly among his most important is the Nyquist criteria for stability. His 1923 landmark paper on the subject, "Regeneration Theory," impacted almost every field of science, as did his noise paper.

Noise became a fundamental study in electrical engineering. Also, the sources of noise in the atmosphere and how noise affects transmission became very important for obvious reasons. The search for the sources of noise was a major quest. For example, weather, noise due to the atmosphere, the sun, etc. In the late 1940s, Bell Laboratories undertook an effort to find and understand all the sources of noise. They built a big antenna and systematically pointed it in different directions to find the sources of noise. However, there was a persistent low noise level, no matter where the antenna was pointed. Since the noise was always there, the first guess was that it was instrumental. Everything possible was done to remove it from the electronics, to no avail. Now, at that time there were two competing theories of the universe. Hubble had discovered that galaxies were moving away from each other and that the further they are, the faster they were moving away. Gamow and others came up with the idea that, indeed, the universe is expanding, and he and others worked out the consequences. Gamow predicted that, in the beginning, the whole universe was the size of a basketball or football or something like that, and it exploded pretty dramatically, as any one might well imagine if we pack the whole universe to the size of a basketball. Also, it was pretty hot, and hence there was a lot of light in the beginning. Now, some 15 billion years later, it is pretty cool, much like how a gas cools when it expands. Remember that all bodies radiate and that radiation is called blackbody radiation and is temperature dependent. Gamow estimated that the radiation now was about 10 K.

The competing theory of the origin of the universe was that of Hoyle, Bondi, and Gold, who figured that the universe must have always been around and pretty much in the same form it is now, in a "steady state." Yet, not quite in steady state since they could not deny the expansion of the galaxies; hence, the density would have to be decreasing. To compensate for the fact that galaxies are moving away from each other and the density is decreasing, they argued that matter must be created in such a way to keep the density the same. Of course, in their model there should be no leftover radiation. In what would turn out to be one of the greatest ironies of all time, to show how preposterous the Gamow model was, Hoyle called it the "Big Bang" to deride it. We should say that Hoyle was a great astronomer and was one of the astronomers who came up with the mind-boggling idea that all atoms except hydrogen and helium were formed in the center of stars. Therefore, keep this in mind: all of us came from a star. Well, at least our atoms did.

Anyway, Dickie, a prominent physicist at nearby Princeton University, was working out how to find and measure the leftover radiation as well as how one could find the funds to do it. Penzias and Wilson, who were working on the Bell project, went to see Dickie for advice on a possible explanation for the isotropic noise they were measuring. Dickie immediately realized what they had discovered, and the rest is history. Everyone now accepts the expansion of the universe. The leftover noise that permeates the universe is equivalent to a source at 3 K, and that is why it's called three-degree blackbody radiation. The fact that this noise exists immediately put an end to the steady-state theory of the universe, much like noise had put an end to the anti-atomists some 50 years earlier. However, the phrase Big Bang, the term of derision, has stuck.

Stochastic processes is a bread and butter subject in electrical engineering for various reasons. In addition to the ones mentioned above, another important reason is that one must know how noise is transformed (propagated) by devices and processes. This is a fundamental issue in probability theory, and electrical engineers have developed it to very high levels.

THE MATHEMATICS OF NOISE

The mathematics of noise was invented by Bachelier, Einstein, Smoluchowski, and Langevin. This occurred within a period of six or seven years, and all worked independently of each other, except for the case of Langevin, who knew Einstein and his work well; indeed, his paper was motivated by Einstein's result. These authors initiated a major mathematical development that continues to this day and that has been applied to an incredible variety of important problems in almost every field of science and engineering with immense success. The field was further developed over the next 50 years by a number of scientists and mathematicians who have come up with the mathematical methodologies that are standard today. Here, we just mention some of the main ideas and scientists. We have already discussed the contributions of Einstein, Langevin, and Bachelier.

SMOLUCHOWSKI

Smoluchowski did almost everything Einstein did, and more. He worked out Brownian motion from both the physics and mathematical points of view, and he also considered many issues that Einstein did not. For example, the behavior of the particle when there is an external force such as the force of gravity, the concept of a transition probability, among other standard issues. Smoluchowski is pretty well known to physicists and chemists who do stochastic processes, but he is rarely mentioned in other fields. To give Smoluchowski any justice in a few pages is impossible. In the words of Chandrasekhar: "The theory of density fluctuations as developed by Smoluchowski represents one of the most outstanding achievements . . . ." and while he "is chiefly remembered as the originator (along with Einstein) of the theory of Brownian motion . . . his role as the founder of the present flourishing discipline of stochastic theory is not." Smoluchowski would be a household name, scientifically speaking, if it wasn't for the overshadowing by Einstein. (This is similar to the case of Robert Hook, one of the greatest scientists of all time, who made major discoveries ranging from biology to physics to construction of microscopes. He was overshadowed by Newton! When Newton said "If I have seen further than others, it is by standing upon the shoulders of giants," it was in a letter to Hook; the implication being that Hook was indeed one of those giants. However, many have speculated that it was a comical remark since Hook was small and a hunchback. Nonetheless, Hook was a giant and so was Smoluchowski, both overshadowed.)

ORNSTEIN, WANG, UHLENBECK, FURTH, CHANDRASEKHAR, KRAMERS, RICE, AND OTHERS

These authors wrote the fundamental mathematical papers on the subject in the 50-year period after the initial development by Einstein, Bachelier, Smoluchowski, and Langevin. The papers of these authors are classic. They are remarkably well written, clear, and direct. They are many orders of magnitude better to read than the numerous books on stochastic differential equations that have appeared over the last 50 years. These authors were not concerned with esoteric issues but with the development of important mathematical ideas and methods.

It is worthwhile to crystalize the two different approaches that have been developed, the relation between the two, and their advantages and disadvantages. The methods are most commonly called the Langevin approach and the Fokker-Planck approach.

THE LANGEVIN APPROACH

The basis of the method is, of course, Langevin's idea that one starts with Newton's law for the situation and puts in the stochastic force as if it was like any other force. One then imposes the statistical properties of the random force. We can best illustrate with what is now called the Ornstein-Uhlenbeck process. Take

$$\frac{du}{dt} + \beta u = A(t) \tag{24}$$

and treat it as an ordinary differential equation with a time-dependent force, $A(t)$. Solving it as an ordinary differential equation, one has

$$u(t) = u(0)e^{-\beta t} + e^{-\beta t}\int_0^t e^{\beta t'}A(t')dt'. \tag{25}$$

Also square,

$$\begin{aligned} u^2(t) =& u^2(0)e^{-2\beta t} + 2u(0)e^{-2\beta t}\int_0^t e^{\beta t'}A(t')dt' \\ &+ e^{-2\beta t}\int_0^t\int_0^t e^{\beta t'+\beta t''}A(t')A(t')dt''dt'. \end{aligned} \tag{26}$$

Everything is deterministic. The stochastic issue comes into play by taking the ensemble average of both sides and imposing the statistical properties for $u^2(0)$, $u(0)A(t)$, $A(t)$, and $A(t')A(t')$. In particular, if one assumes that $u(0)$ and $A(t)$ are not correlated, $A(t)$ is mean zero, and $\langle A(t')A(t'')\rangle = \delta(t' - t'')$, then one gets

$$\langle u\rangle = u(0)e^{-\beta t} \tag{27}$$

$$\langle u^2\rangle = \langle u^2(0)\rangle e^{-2\beta t} + \frac{1}{2\beta}(1 - e^{-2\beta t}). \tag{28}$$

Note that

$$\langle u\rangle_{t\to\infty} = 0 \tag{29}$$

$$\langle u^2\rangle_{t\to\infty} = \frac{1}{2\beta}. \tag{30}$$

These results are very important, and they have been obtained in a few simple steps. But suppose now someone wants $\langle u^4\rangle$, one has to start all over again, that is, raise $u(t)$ in (25) to the fourth power and redo the analysis. Moreover, suppose we want the probability distribution of u and not just the moments. Further suppose we want the expectation values of position, then we have to integrate (25) and again do everything all over again. Further, again, suppose we want the joint probability distribution of position and velocity. How do we do that? So the basic idea here is that, in the Langevin method, one gets specific moments quickly and easily. Moreover, the Langevin equation is transparent because it is Newton's equation. Hence, we can easily verbalize the forces involved. However, the Langevin method does not give you the probability distribution. Of course, if one were to carry out the calculation of $\langle u^n\rangle$ for all n, that is, if one were to explicitly calculate all the moments, then of course the distribution could be obtained in principle. This is the case since, generally, the moments of a distribution determine the distribution. (There are some exceptions, but that's another story.)

FOKKER-PLANCK AND RELATED METHODS

The aim of the Fokker-Planck method is to write a differential equation for the probability distribution as it evolves in time. The probability distribution can be for position, velocity, or both, or for other variables, depending on the problem. For example, the Fokker-Planck equation for the Wiener process is the Einstein equation, (3). For the Ornstein-Uhlenbeck process, (20), the Fokker-Planck equation is

$$\frac{\partial P(x,t)}{\partial t} = \gamma\frac{\partial}{\partial x}xP(x,t) + D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{31}$$

and the impulse response solution is

$$\begin{aligned} P(x,t|x',t') =& \frac{\sqrt{\gamma}}{\sqrt{2\pi D\left(1 - e^{-2\gamma(t-t')}\right)}} \\ & \exp\left[-\frac{\gamma\left(x - x'e^{-\gamma(t-t')}\right)^2}{2D\left(1 - e^{-2\gamma(t-t')}\right)}\right]. \end{aligned} \tag{32}$$

This is the probability impulse-response function or Green's function for a particle "obeying" the Ornstein-Uhlenbeck process, (24). How to go from a Langevin equation to a Fokker-Planck equation is a major mathematical development that is still ongoing, particularly for issues of quantum noise. There is a whole set of methods and ideas associated with this development, such as the Kramers-Moyal, Chapman-Enskog expansions, and master equations, among other ideas. All fall under the general umbrella of obtaining the equation of evolution for the probability density. Of course, the Boltzmann equation is one such equation. Besides the names mentioned, we point out that Kolmogorov gave a general approach for obtaining such equations when the process is Markovian.

PEARSON, RAYLEIGH, KAC, AND RANDOM WALKS

Pearson was a major figure in statistics and, in fact, can be considered as one of its originators. He got involved in all applications and also in many fights with the other major statistician, Fisher. One of Pearson's interests involved the statistics of migration of "populations," that is, attributes, and it was because of this fact that he came up with the random walk problem: "A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these stretches he is at a distance between r and $r + dr$ from his starting point O." This appeared in an article in *Nature* in 1905 [21]. Rayleigh noticed that the mathematics of this problem were identical to another problem he solved 25 years earlier and gave the answer for large r [22]. I am not sure who was the first to make a specific connection between the random walk and Brownian motion, but I believe it was Smoluchowski. Many years later, Kac [23] wrote a very important paper on the random walk. He considered random walks with forces and also with absorbing barriers. He formulated the discrete problem in terms of Markov chains and showed how one can go to the continuous limit to obtain the standard Fokker-Planck type equations.

S.O. RICE

Rice's classical paper, "Mathematical Analysis of Random Noise," which he divided into four parts and published in the *Bell System Technical Journal*, is the foundation and methodology for modern noise calculations [24]. The paper is sweeping, consisting of 162 pages and addressing almost all aspects of noise. It was published in two consecutive volumes in 1944 and 1945. Not only did Rice prove many fundamental results, but he also devised the methods for how noise gets transformed by an electrical device; this is of fundamental importance in physics and engineering.

[THE STORY OF NOISE IS AN EXCITING STORY, FILLED WITH DRAMA, AND WORTH TELLING.]

THE MATCHED FILTER

The matched filter concept was invented by Van Fleck and Middleton, and independently by D. North. It aims at detecting a signal after noise has been added, which of course is the case when a signal propagates. The method was invented during the Second World War. Van Fleck was one of the great physicists of the last century, and Middleton, also a physicist, made many important contributions. Middleton's book *Introduction to Statistical Communication Theory* is one of those texts that is so extraordinary for its clarity and depth that one marvels at it and the author. It is perhaps the greatest book ever written on noise, probability theory, and stochastic processes. The idea of the matched filter is that, in some optimal sense, the best detector is obtained by correlating the signal at hand with the signal that we suspect is in the noise. Van Fleck and Middleton published the method in a report at the Harvard Radio Research Laboratory, and North published it as a report at RCA Laboratories; both were published around 1943 and both were secret reports. Now, the matched filter is a routine detection method in radar and sonar, among many other fields.

FUNDAMENTAL DERIVATIONS OF NOISE

Since we know the governing equations for the motion of atoms or light, it has always been a major aim in science to try to understand and derive the macroscopic probabilistic equations, whether it is the Langevin equation or Fokker-Plank equation from the fundamental equations of motion, be it Newton's laws or the Schrödinger equation. For example, can one derive the Langevin equation from the motions of the atoms and show that the random term is indeed white noise? Many have given such fundamental derivations, and it is still an active research problem, particularly for the quantum case. Of course, the whole concept of trying to understand the macroscopic issues of statistical mechanics from microscopic considerations originated with Boltzmann and others, as we have discussed. Ehrenfest's famous little book called *The Conceptual Foundations of the Statistical Approach to Mechanics*, published in 1912, crystallized this aim [25]. Perhaps the first work to derive the Langevin equation is the well-known paper by Ford, Kac, and Mazur. The recent papers by O'Connell and Ford gives a general model for deriving Langevin-type equations, both in the classical and quantum cases.

BOOKS

Incidentally, Middleton's book is one of the green books published by McGraw-Hill in the series titled *International Series in Pure and Applied Physics*, which includes legendary books by many authors of the highest scientific caliber, including Slater, Straton, Morse and Feshbach, and Kennard. The series was edited by Schiff, the author of the classic textbook on quantum mechanics. Also, McGraw-Hill had another illustrious series titled *Electrical and Electron Engineering Series*. For those of us who remember those books and books like them from different publishers, I think it is worthwhile to mention that these texts brought forth emotions not typical of today's books. Depending on the individual, they could be inspiring or depressing, or both. Inspiring because they were monumental in every way, but mostly because they were able to convey ideas simply. Depressing because one cannot understand how anyone can absorb a subject with such clarity and be able to describe that knowledge with such succinctness and simplicity. These types of books are still being written, but today it often seems that the merit of a book or paper is how many backward epsilons and upside down "A"s it has. Being more than a little familiar with $\in$ and $\forall$ because I studied symbolic logic when I was young, I wonder how it is that so many classic papers do not use them. How did the world invent the fundamental equations of nature, Maxwell's equation, Newtonian mechanics, quantum mechanics, semiconductors, and the laser without using any $\in$ and $\forall$? I do not think any Nobel Prize winner ever published a paper that used $\in$ and $\forall$. I know a very famous scientist/mathematician who says he will not read any paper that contains $\in$ and $\forall$. In fact, no less then the world's authority on mathematical notation, Knuth—the man that brought us TeX—says [26] (with Larabee and Roberts): "Don't use the symbols . . . $\in$, $\forall$ . . . ; replace them by corresponding words. Except in works on logic, of course."

NOISE IN ASTRONOMY

The study of noise is basic in astronomy because it touches almost every aspect of the field both theoretically and experimentally. It would take a whole book to describe what astronomers have contributed, and we can not do it any justice here. It is without question that the most important and interesting review article on noise was written by one of the greatest astronomers of all time, Chandrasekhar, who also was certainly one of the greatest scientific writers of all time [27]. Chandrasekhar wrote many articles on noise, and there is a series of papers by him and von Neumann on the fluctuations of the gravitational force, where many new ideas about noise were developed [28], [29]. My own interest in the field came from studying these classic papers, and they motivated me to verify some of the results by computer simulations.

One of the reasons noise is so important in astronomy is that one is always studying fluctuations, for example, in the light output, in scattering, in density. Also, often in astronomy, physical quantities that are being measured have such a low intensity that fluctuations or noise become increasingly important. However, an equally important historical reason is that of "spectral lines." The reason we have discovered so much about the universe is that atoms and molecules can be identified by their spectral lines—spectral lines fingerprint atoms and molecules. This was one of the most important discoveries of all time, and it is what opened modern science and engineering. The discovery that spectral lines can be used as identifiers of atoms and molecules was made by Bunsen and Kirchoff around 1850 and immediately become the most powerful method for discovering what atoms or molecules exist in a piece of paper, or on a planet, or on a star. The reason for, and the understanding of, why spectral lines are unique to each atom and molecule did not come about until the discovery of the quantum mechanical laws of nature. But the important thing for our consideration is that spectral lines have widths, and the widths are an indication of the environmental parameters, such as temperature and pressure. But spectral line widths are fluctuations, that is, noise. Typically, the shape of a line can be Gaussian or Lorenzian. The mathematical development for the study of these widths, and the discovery of the physical mechanisms producing the widths, has been an active area of research for over 100 years, with major contributions by astronomers.

WIENER AND PERRIN: PATHOLOGICAL FUNCTIONS AND THE WIENER PROCESS

We will soon give a short history of "pathological" functions, functions that are continuous everywhere but nowhere differentiable. These are the functions that Poincaré said "will never have … use" and that Hermite called "a plague." It is fascinating that it was Perrin, the experimentalist, who first speculated that perhaps Brownian paths are continuous, but nowhere differentiable. And it was Perrin who inspired Wiener to develop idealized Brownian motion paths and prove mathematically that, indeed, they are continuous but nowhere differentiable. To quote Wiener: "The Brownian motion . . . . There were fundamental papers by Einstein and Smoluchowski that covered it, but whereas these papers concerned what was happening to any given particle at a specific time, or the long-time statistics of many particles, they did not concern themselves with the mathematical properties of the curve followed by a single particle. Here the literature was very scant, but it did include a telling comment by the French physicist Perrin in his book *Les Atomes*, where he said in effect that the very irregular curves followed by particles in the Brownian motion led one to think of the supposed continuous non-differentiable curves of the mathematicians. He called the motion continuous because the particles never jump over a gap, and non-differentiable because at no time do they seem to have a well-defined direction of movement."

We quote from Perrins' book: "the apparent mean speed of a grain during a given time varies in the wildest way in magnitude and direction and does not tend to a limit as the time taken for an observation decreases" and "nature contains suggestions of non-differentiable as well as differentiable processes." Wiener [30] picked up the idea and developed the mathematics to the satisfaction of some mathematicians. So here we are around 1915–1925, about 50 years after the beginning of the "growing plague," the construction of pathological functions, an experimentalist, Perrin, finally says maybe there is something to them!

Try to imagine a function that is continuous but nowhere differentiable and perhaps you will agree with Hermite, the great astronomer and mathematician: "I turn away in horror and disgust from the growing plague of nondifferentiable functions." The great Poincaré said "Yesterday, if a new function was invented it was to serve some practical end; today they are specially invented only to show up the arguments of our fathers, and they will never have any other use." Nonetheless, the development of these pathological functions continued with major contributors such as Cantor, who brought us the super infinite, the idea that there are regular infinities and then there are super infinities. In particular, the infinity of integers is the same as the infinity of fractions, but the infinity of irrational numbers is greater than the infinity of the integers or fractions. Then there was Peano, who brought as the formalization of mathematics and space-filling curves, as well as many others such as Koch. Every school child now gets a math project to draw Koch curves and, of course, we now have the "fractals are everywhere" movement.

To give an idea why all this was so dramatic, we briefly review the concept of a function, culminating with Weierstrass's construction of a nowhere differentiable but continuous function. The concept of a function has a long history and has been one of the main themes in mathematics for about 250 years. Differentiability also has a long and deep history, and of course the two are intertwined. We all learn that a function can be continuous but not differentiable, and from a simple viewpoint all that means is that a function can have a kink at a point and hence the tangent to the curve is not uniquely defined. The modern concept of a function, the idea that to an independent variable we associate a value, began to develop in the mid-1700s with a major controversy of the discoverers of the wave equation, Euler and d'Alembert. d'Alembert said a function must be smooth if it is to be a solution of the wave equation, but Euler said it can have kinks and still be an initial condition to the solution to the wave equation; after all, that is how a violin string is plucked. d'Alembert argued that it clearly could not be so because the wave equation has a second derivative. Incidentally, Bernoulli also jumped in and basically gave the beginnings of what we now call Fourier analysis. This controversy started it, but it is historically clear that the issue and controversy of "what is a function?" achieved great intensity with Fourier.

FOURIER, HEAT, AND JUMPS

Fourier was a physicist and solved the main problem of his time. By the time Fourier was around, the wave equation had already been discovered and the problem of the century, so to speak, was the nature of heat, and getting the heat equation. Fourier got it,

$$\frac{\partial T(x,t)}{\partial t} = \kappa \frac{\partial^2 T(x,t)}{\partial x^2}, \tag{33}$$

where $T(x, t)$ is the temperature at position x at time t. There was no controversy about the equation, and everyone knew that finally the heat equation was discovered. Fourier did this around 1800, and his famous book called *The Theory of Heat* was published in 1822. The book is available from Dover for a few dollars.

However, along the way, Fourier found methods of solution that were more than unusual. He claimed that *any* function, particularly functions that have jumps, defined in the interval $[-L, L]$ can be expressed as

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x. \tag{34}$$

This was an apparently absurd claim, and many of the greats of that time thought that it was more than preposterous. The reason they thought so is that since sine and cosine are analytic continuous functions and do not jump in value, the sum should also not jump in value! The reason it is important that any function be expandable is due to the following typical heat problem. Think of an object at a certain temperature and put it in contact with another object at a different temperature. How does the temperature evolve in time at every point in both objects? The heat equation allows one to solve that problem. But what are the initial conditions? At the interface, the objects are at different temperatures and, hence, there is a jump at a point as one goes from one object to another. Fourier analysis enables the handling of a jump!

These issues led to the fundamental question: What functions have a Fourier expansion? Dirichlet was the first to crystalize the issue by giving sufficient conditions for a function to have a Fourier expansion. Dirichlet himself then invented a function that has no Fourier expansion and, moreover, does not seem to have an analytic expression in the sense we are all used to. The Dirichlet function is that $f(x) = 0$ when x is rational and $f(x) = 1$ when it is irrational. Try to draw it. So Dirichlet invented a nutty function and showed that it has no Fourier representation, but surely we are interested in smooth functions, whatever that means. One had the sense that for continuous functions, everything is more or less all right except maybe at a few points. But a few years later, around 1870, Karl Weierstrass startled everyone who cares about such things by constructing a function that is everywhere continuous but has a kink at every point. That is, it is continuous everywhere but differentiable nowhere! For those who have never seen the famous Weierstrass function, it is

$$f(x) = \sum_{n=1}^{\infty} b^n \cos a^n \pi x, \tag{35}$$

where b is any number between zero and one, a is an odd integer, and the product ab is so chosen that $ab > 1 + 3\pi/2$. In the same year, Heine, based on ideas of Weierstrass, gave us what we all loved in calculus, namely, the "for every epsilon there is a delta" definition of a limit.

So now we have continuous curves being kinky in every way. Every point is a kink! This caused quite a stir among certain mathematicians, and debates ensued as to whether these functions are important or just pathologies not worthy of study. Some loved it and some hated it. The construction of such functions continued, and are now sometimes referred to as "pathological functions." All kinds of pathological functions were constructed, including space-filling curves.

While all this was very interesting to mathematicians who cared about such things, it is certainly the case that for the first 50 years or so after Weierstrass, no scientist took these functions seriously in the sense that anything in nature would be so pathological. There was no discussion that these functions represented anything in nature or would be useful to represent anything in nature. So it is that much more surprising that Perrin would be the one to suggest that such functions may indeed represent some aspects of the real world!

WIENER

Wiener was one of the last traditional great mathematicians who knew and contributed to physics and engineering. Indeed, it can be argued that he was one of the founders of modern electrical engineering, having made very basic and important contributions to control theory, filtering, stochastic processes, and noise. He was a child prodigy who had great ambitions. Wiener realized that there is rich mathematics behind the Einstein-Smoluchowski formulation of Brownian motion. The main issue that attracted him was defining Brownian paths from a mathematical point of view. The inherent difficulty and the mathematical interest is that, as Perrin pointed out, while the paths are continuous, they can suddenly change directions and hence are not differentiable. Another way to look at it is that the standard deviation goes as the square root of time, and hence the standard deviation divided by time goes as one over the square root of time and diverges for time going to zero. So Wiener formulated Brownian motion paths armed with what was in the air at that time, measure theory. He formalized and defined, from a mathematical viewpoint, a probabilistic process that was an idealization of Brownian motion as formulated by Einstein and others [30]. This was in 1923. It is a gross mistake to claim, as many books do, that Wiener developed the mathematics of Brownian motion. Wiener's true contribution was in defining "measure" for Brownian paths. I know from experience that most people fall asleep when one starts to talk about "measure," but let us just say that probability theory in the last 80 years or so has been formulated in terms of measure and sets and stuff like that. In fact, probability is reduced to an axiomatic formulation in terms of spaces and sets, something that is appealing to some. It is commonly said in the mathematical literature, and even sometimes in the engineering literature, that Wiener showed that Brownian motion exists! What is meant is that, given Wiener's mathematical idealization, one can prove that the process "exists" in some mathematical sense.

The "Wiener process" has come to be defined as the "solution" of the differential equation

$$\frac{dW(t)}{dt} = F(t), \tag{36}$$

where $F(t)$ is white noise. That is, $F(t)$ is a random function that has mean zero and $\overline{F(t)F(t')} = \delta(t - t')$. Also, one can define the process by saying that a) $W(t)$ is mean zero, b) the variance of $W(t) - W(t') = t - t'$, and c) If $t_1 < t_2 \ldots < t_n$, then $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are mutually independent Gaussian variables. But note that (36) is just a special case of the Langevin equation, and its basic properties have been developed by the mathematicians and scientists mentioned in the section "The Mathematics of Noise." Its quite clear that if one wants to associate a name with the fundamental results of (36), it should be Bachelier or Einstein or Smoluchowski. In fact, sometimes this is done even in the mathematical literature. It is a mistaken view to say that Wiener developed the properties of Brownian motion. What he did was formulate a measure description for the paths. If you don't know what that means, that's fine. Most people who misuse the phrase "Wiener process" don't either.

THE TERMINOLOGY "BROWNIAN MOTION"

Unfortunately, there currently are two uses of the phrase "Brownian motion": the physical phenomena of the scientists and the "mathematical" Brownian motion as defined by Wiener and others. I can think of nothing worse than this state of affairs because it implies a gross misrepresentation of history. The mathematics of Brownian motion were developed by Einstein, Smoluchowski, Bachelier, and all the others mentioned in the section "Mathematics of Noise." As we said above, Wiener, as well as others whom we discuss in the next section, made an important contribution from a certain mathematical perspective that is not necessarily of interest to all. It is not uncommon in the mathematical literature to simply give a mathematical definition of noise or Brownian motion in terms of spaces and sets and then state that "Wiener proved it exists," never mentioning the monumental works of the great mathematicians we discussed above, starting with no less a figure than Einstein.

THE "EMBARRASSMENT" AND ITS SOLUTION: DOOB, ITÔ, AND STRATONOVICH

The mathematics of noise as developed by the scientists and mathematicians mentioned in the section "The Mathematics of Noise" is wonderful, profound, and deep. I emphasize that, in this section, we describe a different kind of mathematical motive, and I leave it to the reader to label it. One must never forget the Wang-Uhlenbeck admonition that appeared in a footnote of their classic 1945 paper on Brownian motion [31]: "The authors are aware that in the mathematical literature . . . the notion of a random (or stochastic) process has been defined in a much more refined way . . . . However it seems to us that these investigations have not helped in the solution of problems of direct physical interest." Every interested reader must decide for himself whether this statement is still true after many more years of mathematical refinement. From a purely mathematical point of view, Norbert Wiener started it, with subsequent contributions by Levy, Kolmogorov, Doob, and others. We already mentioned that Ornstein, Wang, and Uhlenbeck were the first to study in full detail the Langevin equation, which we repeat here for convenience [31], [32]

$$\frac{du(t)}{dt} + \beta u(t) = F(t). \tag{37}$$

Doob [33] made important contributions because he developed certain mathematical issues regarding it. In his own words, "A stochastic differential equation will be introduced in a rigorous way to give precise meaning to the Langevin equation . . . this will avoid the usual embarrassing situation in which . . . the second derivative of $x(t)$ is used to find a solution $x(t)$ not having a second derivative." This is a crucial statement and the first time I read it, over 40 years ago, I didn't get the "embarrassing" situation. First, one must be very careful to understand what "a stochastic differential equation will be introduced" means. What it means is that he will reformulate the Langevin equation so that one can study its properties in a rigorous way. As to "avoid the usual embarrassing situation in which . . . the second derivative of $x(t)$ is used to find a solution $x(t)$ not having a second derivative," what he means is the following: First, of course, since $u = (dx(t))/dt$, then $(du(t))/dt = (dx^2(t))/dt^2$. His point of embarrassment is that we know that the paths have kinks so the derivative doesn't exist, but nonetheless we write an equation that has a nonexistent derivative and then try to solve it to get a quantity that does exist. There have been many such "embarrassments" in the history of mathematics. To give a modern example, many pure mathematicians laughed at Heaviside for his operational calculus and also laughed at the delta function. But whether one should be embarrassed by such a powerful method is a question of taste.

The problem, to quote Doob, "is to find a proper stochastic analog of the Langevin equation, remembering that we do not expect $u'(t)$ to exist." What Doob did was rewrite the Langevin equation in the form

$$du(t) = -\beta u(t)dt + dF(t). \tag{38}$$

He aimed to "give these differentials a suitable interpretation." So he developed the properties of the differentials, which basically means the differences, and never had to consider derivatives in the usual sense.

It is a general consensus among certain mathematicians that Itô opened the modern thinking of stochastic differential equations [34]. First, one considers the more general type of Langevin equation

$$\frac{du(t)}{dt} = a(u(t), t) + b(u(t), t)F(t). \tag{39}$$

Since we have no idea how to define differentiability when it comes to a random process, Itô says, instead of (39), convert it to

$$u(t) - u(0) = \int_0^t a(u(t'), t')dt' + \int_0^t b(u(t'), t')F(t')dt'. \tag{40}$$

Of course, writing (40) is what is typically done in the study of the Langevin equation, as described in the "The Mathematics of Noise" section. However, now we transform the mathematical issues from worrying about how to define differentiability of a stochastic variable to defining the integral of a process. The idea here is that if we can give a sensible definition of an integral of a stochastic process, we will then have a rigorous mathematical description and avoid issues of differentiation. Itô defines a general stochastic integral by

$$\lim_{N\to\infty} \int_0^t B(t')dF(t')dt' = \lim_{N\to\infty} \sum_{i=1}^{N} B(t_{i-1}) \times [F(t_i) - F(t_{i-1})]. \tag{41}$$

This allows one to develop the mathematics in a consistent fashion because we have defined what a stochastic integral is. Moreover, and crucially important, one can manipulate stochastic differentials in a consistent manner, much as we are used to manipulating ordinary differentials in calculus. For example, one can make a change of variables and get a new stochastic process, chain rule, etc. The calculus thus developed has a number of strange properties, strange only in the sense that it does not obey the usual rules of differentials of ordinary calculus.

Stratonovich defined the stochastic integral in a different way, namely by [35]

$$\int_0^t B(t')dF(t')dt' = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{2}[B(t_i) + B(t_{i-1})] \times [F(t_i) - F(t_{i-1})], \tag{42}$$

and showed that one then gets rules of manipulation that are closer to the rules of ordinary calculus.

Of course, if the quantities B and F were ordinary functions, one would get the same answers for the Itô and Stratonovich integrals. It is remarkable that such similar definitions for the integrals can produce very different results when B and F are random processes.

QUANTUM MECHANICS, QUANTUM NOISE, AND THE LASER

Since quantum mechanics is so different than classical mechanics, and since quantum mechanics is inherently probabilistic, one would expect it would produce a new kind of noise. Indeed, that is the case. The most important and interesting developments in noise theory and applications over the last 50 years have come from quantum mechanics, specifically, that which is called quantum noise. It is full of new ideas, new phenomena, new methods, and new applications, all undreamed of in the standard formulation of noise and probability.

Quantum mechanics is not only the most successful theory ever devised, but the panorama of successes—from atoms to molecules, to solids, to liquids, to transistors, to lasers, to explaining the age old riddle of the source of the sun's energy and light, and numerous other phenomena—stagers the imagination. Newton's equations have been replaced by the Schrödinger/Heisenberg equation of motion. However, besides its successes, quantum mechanics has totally changed our view of the world because it is an inherently probabilistic theory. It is curious enough that the basic workings of nature are probabilistic, but what is also curious is the type of probability theory it is—totally different than standard probability theory devised over the last 300 years. The fact that quantum mechanics is so successful and yet so different than standard probability theory is a great mystery that has not yet been explained. Obviously, whoever created the rules of this universe did not read the classic book by Feller or the axiomatic formulation of Kolmogorov (or did read them and thought they weren't appropriate for this universe).

The reason quantum mechanics is so strange as a probability theory is because one deals with operators and state vectors/wave functions instead of classical functions, and yet one ends up with measurable quantities that are probabilities and expectation values. We also point out that, in quantum statistical mechanics, two sorts of probabilities exist: the inherent one obtained from the wave function and also one that has to do with "ignorance," namely situations where we do not know the wave function but can assign probabilities for having a variety of different ones. This second probability is analogous to the probability in classical statistical mechanics. Hence, in general, two probabilistic averages are done in such cases. Perhaps an analogy is appropriate. Suppose we have two dice, a and b, and suppose one of them is fair but the other has probabilities 1/2 to get the number six and 1/10 for each of the remaining numbers. Each die is an independent system, and we can calculate any probability quantity we want for each die. So, for example, the expectation value of the first die is 3.5 and that of the second is 4.5. Let us call those two expectation values $< a >$ and $< b >$. But suppose that we do not know which die is going to be thrown or suppose we throw a die infinitely often, but which die we throw is given probabilistically. Let us call the probabilities of choosing a particular die p_a and p_b. Now the expectation value for the "system" is

$$\langle \text{system} \rangle =< a > p_a + < b > p_b. \tag{43}$$

So, for example, if $p_a = p_b = .5$, then the expected value would be four. In the quantum case, the calculation of the averages $< a >$and $< b >$ is done using the quantum mechanical wave function and operators, but the second calculation, (43), is the standard probability method. To handle both probabilities in a unified way, Dirac and von Newman devised a mathematical construct called the density operator or density matrix.

EINSTEIN'S AIM WAS TO FIND A MEASURABLE EXPERIMENTAL MANIFESTATION OF ATOMS, TO PREDICT A MACROSCOPIC OBSERVABLE FACT.

The importance of quantum mechanics to the understanding of noise was addressed since the early days of quantum mechanics and modern electronics. Moreover, the fundamental calculations of Planck and Einstein on blackbody radiation can be viewed as noise calculations. Also, as semiconductors became important, the understanding of noise in them became extremely important, and quantum mechanics must be used to understand semiconductors and the inherent noise they produce. But starting with the invention of the maser and laser and the field of quantum optics, quantum noise has been developed to a very high level. The issues, both physical and mathematical, are challenging and fascinating, and many fundamental problems remain unsolved. Issues such as the quantum Langevin equation and quantum Fokker-Planck equations are active areas of research. Also, we mention that the vacuum according to quantum mechanics, is not a "vacuum," but is always fluctuating and is a random process! Here's one way to think about it: Place one atom, let us take the simplest, a hydrogen atom, in the traditional vacuum; then, according to elementary quantum mechanics, the potential on the electron is due to electromagnetic force due to the proton. Solving the hydrogen atom quantum mechanically and verifying the theoretical results experimentally was one of the great achievements of quantum mechanics, and it was originally done by Schrödinger. The comparison of experiment and theory is outstanding in every way. However, more refined experiments, such as the Lamb shift, do not agree with theory if only the forces mentioned are taken into account. However, if one assumes that the vacuum is not a "vacuum" but is producing a fluctuating random electromagnetic force, then the agreement with experiment is achieved. We emphasize, though, that the fluctuating random force is not something imposed, but it comes out of quantizing Maxwell's equations.

The invention of the laser initiated an intense examination of quantum noise. The laser produces light of narrow bandwidth around a particular frequency, and it was crucial to understand the bandwidth from a fundamental point of view, if for no other reason than to learn how to control it and make better lasers. The bandwidth of any spectral line is due to fluctuations or noise; otherwise, one would get just a line. Traditionally, most calculations involving atoms and electromagnetic fields were done by treating the atom quantum mechanically, but treating the light as a classical electromagnetic field interacting with it. It became clear that this approximation was not good enough to understand the laser bandwidth and that one must treat both the atom and light in a full quantum mechanical way. Three approaches were developed. Scully and Lamb used a density matrix approach, Lax used the Langevin equation approach, and Haken and Risen used the Fokker-Planck method. We have already discussed the Langevin and Fokker-Planck methods, but we point out that for the quantum case, writing such equations is not straightforward, as one has to deal with noise operators rather than with classical random functions. It is worthwhile at this juncture to discuss the density matrix approach, the fundamental idea of which is best explained in the classical context. Suppose that we have the usual room full of molecules, let's say 10^{23} of them. Now each one obeys Newton's laws and, hence, we have 10^{23} coupled, second-order differential equations. But we really don't care about each molecule because what we want is properties like the distribution of velocity, the density of the gas, that is, the macroscopic quantities. One can write the governing equation for the probability distribution function for the 10^{23} particles and that is called Liouville's equation, which is equivalent to Newton's equations. Since we want to obtain the probability distribution for one molecule, a typical molecule, one integrates out from the Liouville equation one variable, then another, and another, until $10^{23}-1$ variables are integrated out. This results in governing equations for the reduced probabilities, but they are all coupled. The idea is to uncouple them through various approximations that depend on the physical situation. In the classical case, this is called the BBGKY hierarchy, named after Born, Bogoliubov, Green, Kirkwood, and Yvon. In quantum mechanics, one uses the density matrix, and the same idea we just described for classical mechanics holds. The crucial issue of course is to be able to formulate the problem for the full case, that is, all the possible states of the atom and the possible states of the light. In addition, it is crucial to know what approximations to make to decouple the equations.

In relation to quantum noise, there are many unsolved issues, such as getting a proper quantum mechanical Langevin equation, defining autocorrelation functions for operators, and obtaining quantum mechanical Fokker-Planck equations, among many other issues of a fundamental nature. Of particular interest is the generalization of the Nyquist result that we discussed in the section titled "Shot and Thermal Noise: Schottky, Johnson, and Nyquist." Recall that Nyquist took kT per degree of freedom to obtain the noise spectrum, but as we mentioned previously, Nyquist noted at the end of his article that if one uses Planck's law for blackbody radiation, it modifies his formula as indicated by (23). In 1951, Callen and Welton wrote a classic paper [36] titled "Irreversibility and Generalized Noise," which generalizes Nyquist's result. While this paper is usually associated with quantum noise, it addresses a much wider issue: "It has frequently been conjectured that the Nyquist relation can be extended to a general class of dissipative systems other than merely electrical systems. Yet to our knowledge, no proof has been given of such a generalization, nor have any criteria been developed for the type of system or the character of the 'forces' to which the generalized Nyquist relation may be applied. The development of such a proof and of such a criteria is the purpose of this paper." For the quantum noise case, they obtained the Nyquist result, (23), but with an extra term for the voltage fluctuation

$$E^2 = \frac{2R}{\pi}\left(\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}\right). \qquad (44)$$

The extra term, $\hbar\omega/2$, remains no matter how one controls the temperature. This term, which is called the zero point noise fluctuation or zero point energy, is responsible, to a large extent, for the richnesses of quantum noise and its manifestations.

CONCLUSION

We hope to have conveyed some of the vibrant history of noise and to have done some justice to a field that has been involved in the solution of some of the greatest scientific, mathematical, and technological problems. It is often said that noise is the study of fluctuations about the average. This does not do any justice to the richness of the field. But, if one studies it a bit and appreciates that it was developed by the greatest of minds, including Einstein, then perhaps one can start to get an appreciation for what the field is all about. We hope we have conveyed some of the past great conceptual innovations and practical accomplishments of the field; "noise" will certainly continue to bring new, powerful, interesting, and dramatic ideas and surprises.

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I. Oppenheim, K.E. Shuler, and G.H. Weiss, *Stochastic Processes in Chemical Physics: The Master Equation.* Cambridge, MA: MIT Press, 1977. A collection of papers on stochastic processes, with an emphasis on the Fokker-Planck equation. There is an excellent introduction to stochastic processes.

P.H. Cootner, *The Random Character of Stock Market Prices*. Cambridge, MA: MIT Press, 1964. The classic papers and Bachelier's thesis are here translated into English. Also, Cootner writes an introduction to different papers that are insightful. In regard to Bachelier, I mention here the excellent article by Courtault et al., "Louis Bachelier on the centenary of *theorie de la speculation, Math. Finan.*, vol. 10, p. 341, 2000 and also "Bachelier and his times, A conversation with Bernard Bru," *Finan. Stoch.*, vol. 5, 3–32, 2001.

R. Lindsay, *Early Concepts of Energy in Atomic Physics*. Stroudsburg, PA: Dowden, Hutchinson Ross, 1979. This volume contains many of the historical papers, including the 1828 paper by Brown. It also has the Langevin paper translated into English.

Books on noise and stochastic processes

There are numerous books, but I list the ones that I think most would agree are classics. (Of course, I should list Feller (both volumes), but I consider them to be on probability rather than noise/stochastic processes).

D. Middleton, *Introduction to Statistical Communication Theory*. New York: McGraw-Hill, 1960. The classic book on noise, written with style and elegance, covers a panoramic view unmatched by any other book. It is over 1,000 pages in length and is the place to learn noise theory and practical applications.

A. Papoulis, *Probability, Random Variables and Stochastic Processes*. New York: McGraw Hill, 1984. Another classic and more than classic in engineering. There are a number of editions, and recently a new one has been brought out with S. Unnikrishna Pillai as coauthor. (Papoulis recently passed away.)

C.W. Gardiner, *Handbook of Stochastic Methods*, New York: Springer, 1983.

Whatever image "handbook" conjures up, leave it aside. This is an outstanding book in every way. It covers a wide range of topics and methods, and it is clear and very well written.

M.O. Scully and M.S. Zubairy, *Quantum Optics*. Cambridge, UK: Cambridge Univ. Press, 1997. A panoramic view of quantum noise from every perspective, foundations and application. The title is deceptive as the text covers much more than traditional quantum optics. It discusses all the wonderful, new, and surprising views of nature that quantum mechanical processes have brought.

J.L. Doob, *Stochastic Processes*. New York: Wiley, 1958. The classic book for mathematicians.

M.S. Bartlett, *Introduction to Stochastic Processes: With Special Reference to Methods and Applications*. Cambridge, UK: Cambridge Univ. Press, 1961.

D.K.C. MacDonald, *Noise and Fluctuations*. New York: Wiley, 1962. Extraordinary! A model of clear writing. See also: "The Brownian motion and spontaneous fluctuations of electricity," *Res. Appl. Indust.*, vol. 1, pp. 194–203, 1948. (Reprinted in Gupta [3].)

N. van Kampen, *Stochastic Processes in Physics and Chemistry*. Amsterdam: North-Holland, 1981. A standard, clearly and elegantly written.

W.B. Davenport, Jr., *Probability and Random Processes: An Introduction for Applied Scientists and Engineers*. New York: McGraw-Hill., 1970. See also: W.B. Davenport, Jr. and W.L. Root, *An Introduction to the Theory of Random Signals and Noise.* New York: McGraw Hill, 1987.

Review articles

There is one review article that is the absolute one and will always remain so for many reasons, not the least of which is the fact that Chandrasekhar was one of the greatest science writers:

S. Chandrasekhar, "Stochastic problems in physics and astronomy," *Rev. Mod Phys.*, (reprinted in Wax), vol. 15, no. 1, pp. 1–89, 1943.

Two others

M.C. Wang and G.E. Uhlenbeck, "On the theory of Brownian motion," *Rev. Mod. Phys.*, (reprinted in Wax), vol. 17, no. 323, pp. 323–342, 1945. The classic place to learn the subject and the best place to learn about the Langevin equation.

J.B. Johnson, "Electronic noise, the first two decades," *IEEE Spectr.*, (Reprinted in Gupta), vol. 8, pp. 42–46, Feb. 1971.

Books on the history of the atom

There are many old books on the history of the atom, but these are relatively recent ones that are outstanding and very enjoyable:

S.G. Brush, *Statistical Physics and the Atomic Theory of Matter*. Princeton, NJ: Princeton Univ. Press, 1983.

D. Lindley, *Boltzmann's Atom: The Great Debate That Launched a Revolution in Physics*. New York: Free Press, 2001.

B. Pullman, *The Atom in the History of Human Thought*. London, UK: Oxford Univ. Press, 1998. Pullman was a major contributor to quantum chemistry. Most of the quotes I give regarding anti-atomists come from this book.

J.M. Nye, *Molecular Reality: A Perspective on the Scientific Work of Jean Perrin*. New York: Elsevier, 1972. A wonderful scientific biography that covers many issues besides Perrin.

Other books

Sometimes one comes across books that are more than just great. Here are three recent ones that are remarkable and make gripping reading:

B.J. Hunt, *The Maxwellians*. Ithaca, NY: Cornell Univ. Press, 1991.

P.J. Nahin, *The Science of Radio*. New York: Springer-Verlag, 2001.

P.J. Nahin, *Oliver Heaviside, Sage in Solitude*. Piscataway, NJ: IEEE Press, 1988.

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